Math 40510, Algebraic Geometry

Problem Set 1, due February 10, 2025

Note: Answers that are sloppy, either from a mathematical point of view or because they are hard to read, will result in points being deducted even if they are technically correct.

Solutions

1. In class on 1/13/25 we defined the polynomial ring $R = k[x_1, \ldots, x_n]$, where k is a field, and we considered

$$[R]_d = \{\text{polynomials of degree } \leq d\} \cup \{0\}.$$

In this problem we'll find the dimension of this vector space.

a) (7 points) Prove (using a suitable induction) that

$$\binom{d-1}{d-1} + \binom{d}{d-1} + \cdots + \binom{n+d-3}{d-1} + \binom{n+d-2}{d-1} = \binom{n+d-1}{d}.$$

Solution: For convenience let's write it in reverse order

$$\left[\binom{d-1}{d-1} + \binom{d}{d-1} + \cdots + \binom{n+d-3}{d-1}\right] + \binom{n+d-2}{d-1}$$

By induction on d, the part in brackets is $\binom{n+d-2}{d}$. Since $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, we have that the above is

$$= \binom{n+d-2}{d} + \binom{n+d-2}{d-1} = \frac{(n+d-2)!}{d!(n-2)!} + \frac{(n+d-2)!}{(d-1)!(n-1)!}$$

Now get a common denominator:

$$= \frac{(n+d-2)!(n-1)}{d!(n-1)!} + \frac{d(n+d-2)!}{d!(n-1)!} = \frac{(n+d-2)!(n-1+d)}{d!(n-1)!}$$

$$= \frac{(n+d-1)!}{d!(n-1)!} = \binom{n+d-1}{d}.$$

b) (7 points) Find the dimension of $[R]_d$.

[Hints:

- The answer should be a single binomial coefficient. You'll lose points if your answer is more complicated, even if what you write is correct.
- In the class notes we gave a basis for the vector space $[R]_d$. If you weren't in class, you can find the notes uploaded to the webpage. Let me know if you can't find it. You can use this as your starting point. I want a rigorous proof of the stated problem, and very likely you'll want to use special cases of part a).]

Solution: It's enough to find how many elements there are in the basis. The basis we gave was in the middle column of the following table, and in the last column we use part a) for different values of d.

degree	elements	total number
0	1	1
1	$x_1, x_2, \ldots, x_n$	$n = \binom{n}{1}$
2	x_1(everything in $x_1, \ldots, x_n$ of degree 1) x_2(everything in $x_2, \ldots, x_n$ of degree 1) x_3(everything in $x_3, \ldots, x_n$ of degree 1) $\vdots$ $x_n(x_n)$	$n + (n-1) + \cdots + 1$ $= \binom{n+1}{2}$
3	x_1(everything in $x_1, \ldots, x_n$ of degree 2) x_2(everything in $x_2, \ldots, x_n$ of degree 2) $\vdots$ x_{n-1}(everything in x_{n-1, x_n} of degree 2) x_n(everything in x_n of degree 2)	$\binom{n+1}{2} + \binom{n}{2} + \ldots \binom{3}{2} + \binom{2}{2}$ $= \binom{n+2}{3}$
$\vdots$	$\vdots$	
d	etc.	$\binom{n+d-2}{d-1} + \binom{n+d-3}{d-1} + \cdots + \binom{d-1}{d-1}$ $= \binom{n+d-1}{d}$

So the total number is the following. We'll use the fact that $\binom{n}{k} = \binom{n}{n-k}$.

$$1 + \binom{n}{1} + \binom{n+1}{2} + \binom{n+2}{3} + \cdots + \binom{n+d-1}{d}$$
$$= \binom{n-1}{n-1} + \binom{n}{n-1} + \binom{n+1}{n-1} + \binom{n+2}{n-1} + \cdots + \binom{n+d-1}{n-1}.$$

Now use a) again, interchanging the roles of n and d. We get

$$\dim[R]_d = \binom{n+d}{n}.$$

c) (7 points) Verify your answer to a) with the case $n = 4,\ d = 3$ by listing all of the elements of the basis and counting, to confirm that your part a) gives the right answer in this case.

Solution: Since $n = 4$ and $d = 3$ we expect our basis to have $\binom{4+3}{4} = 35$ elements. Let's use w, x, y, z as our indeterminates. The basis elements are:

$$1$$
$$w, x, y, z$$
$$w^2, wx, wy, wz, x^2, xy, xz, y^2, yz, z^2$$
$$w^3, w^2x, w^2y, w^2z, wx^2, wxy, wxz, wy^2, wyz, wz^2, x^3, x^2y, x^2z, xy^2, xyz, xz^2, y^3, y^2z, yz^2, z^3$$

and there are indeed 35 of them.

2. We saw in class that if $f, g \in k[x]$, where k is a field, then there are unique elements $q, r \in k[x]$ such that

- $f = qg + r$, and
- either $r = 0$ (as a polynomial) or $\deg r < \deg g$.

In this problem we'll talk about what happens if k is not a field. So consider $\mathbb{Z}_6[x]$.

a) (7 points) Give an example of polynomials f and g in $\mathbb{Z}_6[x]$ so that no q and r exist with the stated properties.

Solution: Of course the condition $\deg r < \deg g$ is important – otherwise we could always take $q = 0$, $g = f$ as a solution.

Let $f = x^5$, $g = 3x^2$. Let

$$q(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

Then

$$qg = 3a_dx^{d+2} + 3a_{d-1}x^{d+1} + \cdots + 3a_4x^6 + 3a_3x^5 + 3a_2x^4 + 3a_1x^3 + 3a_0x^2.$$

If $gq + r = f$ then in particular either $3a_3 = 1$ or r has x^5 a term involving x^5. The first is impossible since 3 has no inverse in $\mathbb{Z}_6$. The second violates the condition $\deg r < \deg g$.

b) (7 points) Give an example of polynomials f and g in $\mathbb{Z}_6[x]$ so that q and r exist but q is not unique.

Solution:
Take $f = 3x^5$, $g = 3x^2$. Then

$$3x^5 = 3x^2(x^3 + 2x^a)$$

In this example $r = 0$ but $q = x^3 + 2x^a$ works for all choices of a, so q is not unique.

3. We saw the following facts in class:

- A finite subset of k^n is an affine variety.
- For any n, k^n is a subvariety of k^n (i.e. k^n is contained in k^n and is an affine varitey), since $k^n = \mathbb{V}(0)$.

For the following problems, assume that k is an infinite field.

a) (7 points) Let X be an infinite proper subset of k^1. Then X is not an affine variety.

Solution:
If X were an affine variety in k^1 then there exist $f_1, \ldots, f_s \in k[x]$ such that $X = \mathbb{V}(f_1, \ldots, f_s)$. On the other hand, for any $i = 1, \ldots, s$ we have $f_i(p) = 0$ for all $P \in X$. But X is infinite, so f_i has infinitely many roots. Thus f_i is the zero polynomial. But this means $X = k^1$, and hence is not a proper subset of k^1.

b) (7 points) Give an example of an infinite proper subset of k^n, for $n \geq 2$, that is an affine variety. Justify your answer.

Solution:
Let X be the set

$$X = \{(t, 0, 0, \ldots, 0) \mid t \in k\}.$$

Then $X = \mathbb{V}(x_2, x_3, \ldots, x_n)$ so it is an affine variety. It is clearly proper, and it is infinite since k is.

c) (8 points) Let

$$X = \{(a, b, c) \mid a, b, c \in \mathbb{Z} \text{ are prime numbers or } 0\}.$$

Prove that X is not an affine variety.

Solution:
Suppose X were an affine variety, so $X = \mathbb{V}(f_1, \ldots, f_s)$.

Fix any i with $1 \le i \le s$. Fix prime numbers b, c and set

$$g_{b,c}(x) = f_i(x, b, c).$$

Now $g_{b,c}(x)$ is a polynomial in one variable that vanishes whenever x is a prime number, by assumption. But the set of prime numbers is infinite, so $g_{b,c}(x)$ has infinitely many roots, and hence is the zero polynomial. Similarly, setting $h_{a,c}(y) = f_i(a, y, c)$ and $\ell_{a,b}(z) = f_i(a, b, z)$, both of these are the zero polynomial.

We want to conclude that f_i is the zero polynomial. Since f_i vanishes at every triple of prime numbers or 0, $f_i(0, 0, 0) = 0$ so the constant term of f_i is 0. Suppose f_i has some non-zero term. It has to involve at least one variable, and without loss of generality say it's x. But we saw $g_{1,1}(x) = f_i(x, 1, 1)$ is the zero polynomial, so the selected term in fact must be zero. Thus f_i is the zero polynomial, for $1 \le i \le s$.

It follows that $\mathbb{V}(f_1, \ldots, f_s)$ must in fact be all of $\mathbb{R}^3$, so the given X is not an affine variety.

4. Consider the curve

$$C = \mathbb{V}(y^2 - 4x^2(x + 2))$$

in $\mathbb{R}^2$. You're welcome to use a graphing program to see what the curve looks like. (It's a nodal cubic.)

a) (7 points) Use the lines through the origin to come up with a parametrization of C the way we did in class on Wednesday (1/22/25). Specifically, give explicit rational functions $a(t), b(t)$ and set $(x, y) = (a(t), b(t))$. [Hint: The first 6 words of this part of the problem are important!]

Solution:
A line through the origin with slope t has equation

$$y = tx.$$

What does the intersection of such a line with C look like? We solve the equations

$$\begin{gathered} y = tx \\ y^2 - 4x^2(x + 2) = 0. \end{gathered}$$

Plugging in tx for y in the second equation we get

$$t^2x^2 - 4x^2(x + 2) = 0$$

so

(1) $$x^2[t^2 - 4x - 8)] = 0$$

So other than at the origin ($x = 0$) the line meets C at a point whose x-coordinate is obtained by setting $t^2 - 4x - 8 = 0$. Thus $4x = t^2 - 8$, so

$$x = \frac{t^2 - 8}{4}.$$

Since $y = tx$, we have our parametrization:

$$(x, y) = \left(\frac{t^2 - 8}{4}, \frac{t^3 - 8t}{4}\right).$$

b) (7 points) Confirm algebraically that for any $t \in \mathbb{R}$, $(a(t), b(t))$ is a point of C.

Solution:
We plug the above parametrization into the equation for C and show that we get 0 (so (x, y) lies on C).

$$\begin{aligned}
y^2 - 4x^2(x+2) &= y^2 - 4x^3 - 8x^2 \\
&= \left(\frac{t^3 - 8t}{4}\right)^2 - 4\left(\frac{t^2-8}{4}\right)^3 - 8\left(\frac{t^2-8}{4}\right)^2 \\
&= \left(\frac{t(t^2 - 8)}{4}\right)^2 - 4\left(\frac{t^2-8}{4}\right)^3 - 8\left(\frac{t^2-8}{4}\right)^2 \\
&= \left(\frac{t^2-8}{4}\right)^2 \left[t^2 - 4 \cdot \frac{t^2-8}{4} - 8\right] \\
&= \left(\frac{t^2-8}{4}\right)^2 \cdot 0 \\
&= 0.
\end{aligned}$$

c) (7 points) Use the geometry to show that for any point P on C other than the origin, there is a **unique** value t_P of t so that $(a(t_P), b(t_P)) = P$.

Solution:
We saw in part a) (equation (1) that the intersection of the line with C consists of a double root when $x = 0$ (which corresponds to the origin) plus a single root that we used for the parametrization. So if P is any point on C other than the origin, there is a unique line through P and the origin, so there is a unique value t_P (giving that unique line) as claimed.

d) (7 points) What are the two values of t that give the origin? Explain.

Solution:
Since our parametrization is

$$(x, y) = \left(\frac{t^2-8}{4}, \frac{t^3-8t}{4}\right) = \left(\frac{t^2-8}{4}, t \cdot \frac{t^2-8}{4}\right),$$

to get $(0, 0)$ we need

$$t = \pm\sqrt{8}.$$

5. a) (7 points) Let $R = k[x_1, \ldots, x_n]$ and let $I_1, I_2, \ldots$ be ideals. (This is not necessarily a finite set of ideals.) Prove that the intersection

$$I = \bigcap_{i \geq 1} I_i$$

is again an ideal.

Solution:

(i) I is not empty since $0 \in I_i$ for all i.

(ii) If $f, g \in I$ then $f, g \in I_i$ for all i, so $f + g \in I_i$ for all i, so I is closed under addition.

(iii) If $f \in I$ and $h \in R$ then $f \in I_i$ for all i, hence $fh \in I_i$ for all i and so $fh \in I$. Thus I is closed under multiplication by an element of the ring.

Therefore I is an ideal.

b) (8 points) Let $R = \mathbb{R}[x, y, z]$. Consider the points $P_i = (i, i^2, i^3) \in \mathbb{R}^3$ for all positive integers i. So for example

$$P_1 = (1, 1, 1), \quad P_2 = (2, 4, 8), \quad P_3 = (3, 9, 27), \ldots.$$

(This is an infinite set.) Find a set of **two** generators for the ideal

$$I = \bigcap_{i \geq 1} I(P_i).$$

In other words, I'm telling you that there exist polynomials F, G so that $I = (F, G)$ and I'm asking you to find F and G. You can use any results from class without justification – just carefully quote the result you're using, and carefully prove that your answer is correct. (If you happen to use an algebra program, it's not enough to run this on the program – you have to justify your answer.)

Solution:

Notice that all of these points lie on the twisted cubic curve $C = \{(t, t^2, t^3) \mid t \in \mathbb{R}\}$. We claim that $I = \mathbb{I}(C) = (y - x^2, z - x^3)$. The fact that these two polynomials generate $\mathbb{I}(C)$ was shown in class on Wednesday, January 29. So it's enough to prove that $I = \mathbb{I}(C)$.

Since $P_i \in C$ for each i, we have $\mathbb{I}(C) \subseteq I$. We have to prove the reverse inclusion. So let $f(x, y, z) \in I$. We want to show that $f(x, y, z) \in \mathbb{I}(C)$.

Use the parametrization. We are assuming that $f(i, i^2, i^3) = 0$ for $i = 1, 2, 3, \ldots$. Consider the polynomial $f(t, t^2, t^3) \in \mathbb{R}[t]$. (This just means you start with the polynomial $f(x, y, z)$ and put in t everywhere you see x, put in t^2 everywhere you see y and put in t^3 everywhere you see z.) Our assumption means that the polynomial $f(t)$ has infinitely many roots, so it must be the zero polynomial. This means $f(t, t^2, t^3) = 0$ for all $t \in \mathbb{R}$, so $f(x, y, z)$ vanishes at every point of C, i.e. $f \in \mathbb{I}(C)$.