1. (10 points) Give an example of a binary operation on \( \mathbb{Z} \) that is not commutative. Explain.

2. Let \( G \) be a group.

   (a) (6 points) Show, as we did in class, that if \( a \in G \) then the inverse element \( a^{-1} \) is unique, i.e. \( a \) has exactly one inverse. (You may use cancelation laws without proof.)

   (b) (7 points) Now assume that \( G \) is a finite group with an even number of elements. Prove that there must be an element \( a \in G \) which is not the identity, but such that \( a^2 = e \). (Hint: part (a) will be helpful. Notice also that \( e \) is its own inverse.)

   (c) (7 points) Show that if \( G \) is a finite group with an odd number of elements then there is no element other than the identity for which \( a^2 = e \). (Hint: Suppose there were, and get a contradiction. Lagrange might be useful.)
3. Consider the permutation

\[ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 7 & 9 & 2 & 8 & 1 & 4 & 6 & 5 & 3 \end{pmatrix} \]

(a) (6 points) Find \( \sigma^2 \) and \( \sigma^{-1} \) write them in the space below:

\[ \sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix} \]

\[ \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix} \]

(b) (4 points) Write \( \sigma \) as a product of disjoint cycles.

(c) (5 points) Write \( \sigma \) as a product of transpositions.
4. In this problem you may assume that we are working in $S_{10}$.

(a) (5 points) Explain why disjoint cycles commute.

(b) (5 points) Let $\sigma_1$ be a cycle of order 6 and $\sigma_2$ be a cycle of order 4. Assume that $\sigma_1$ and $\sigma_2$ are disjoint cycles. What is the order of $\sigma_1\sigma_2$? Explain your answer.

(c) (5 points) Give an example of a cycle of order 6 and a cycle of order 4 (not necessarily disjoint) whose product has order different from the one in part (4b), and say what the order of your product is.

5. (10 points) Find the maximum possible order for an element of $S_8$ and give an example of an element with this order. Explain your answer.
6. (10 points) Above is the group table for the dihedral group $D_4$. Find a subgroup that is isomorphic to the Klein 4-group. Explain. In particular, your answer should include

(a) why it is a subgroup, and

(b) why it is isomorphic to the Klein 4-group.

(Hint: really, the group table on the right is all you need. Giving the group table for your subgroup would be a good approach to both parts, although you should still explain.)
7. Let $G$ be a group and let $H$ be a subgroup of $G$. Assume that $H$ has index 2. (Don’t misread that – it’s index 2, not order 2.)

(a) (5 points) Show that there exists an element of $G$ that’s not in $H$. It’s enough if you show that $H \neq G$. Call such an element $a$, and freely use the element $a$ in subsequent parts of this problem.

(b) (5 points) How many left cosets of $H$ are there? List them. (Here’s where $a$ might come in handy.) Explain your answer.

(c) (5 points) How many right cosets of $H$ are there? List them.

(d) (5 points) Using these answers, show that $aH = Ha$. Explain your answer carefully.