Throughput Capacities and Optimal Resource Allocation in Multiaccess Fading Channels

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Preliminary:

The capacity region for classical discrete memoryless multiple access channel

\[(R_1, R_2) \text{ satisfying: } \]

\[R_1 < I(X_1; Y | X_2)\]
\[R_2 < I(X_2; Y | X_1)\]
\[R_1 + R_2 < I(X_1, X_2; Y)\]

with fixed probability transitions \(p(y | x_1, x_2)\)
and for some independent input distribution \(p(x_1)p(x_2)\)

In the case of the Gaussian multiple access channel \(Y_i = X_{1i} + X_{2i} + Z_i\)

\[R_1 \leq \frac{1}{2} \log(1 + \frac{P_1}{\sigma^2})\]
\[R_2 \leq \frac{1}{2} \log(1 + \frac{P_2}{\sigma^2})\]
\[R_1 + R_2 \leq \frac{1}{2} \log(1 + \frac{P_1 + P_2}{\sigma^2})\]

where \(Z \sim N(0, \sigma^2)\), \(P_1, P_2\) are the power constraint of \(X_1, X_2\)

The upper bounds are achieved when \(X_1 \sim N(0, P_1), X_2 \sim N(0, P_2)\)
Multiaccess Fading Channel Model:

\[ Y(n) = \sum_{i=1}^{M} \sqrt{H_i(n)} X_i(n) + Z(n) \]

\( H_i(n) \) is the fading process of the \( i \) th user, which is jointly stationary and ergodic and whose stationary distribution has continuous density and is bounded.

I. Capacity region of the fixed Gaussian multiaccess channel

\[ C_g(h, P) = \{ R : R(S) \leq \frac{1}{2} \log(1 + \frac{\sum h_i P_i}{\sigma^2}) \text{ for every } S \subset \{1,...,M\} \} \]

II. Capacity region of time-Varying channel with CSI on receiver side only

\[ \{(R_1,...,R_M) : R(S) \leq \mathbb{E}_H \left[ \frac{1}{2} \log(1 + \frac{\sum H_i P_i}{\sigma^2}) \right], \forall S \subset \{1,...,M\} \} \]
Capacity Under Power Control: (CSI on all transceivers)

Def: A power-control policy $PC$ : a mapping of $h = (h_1, \ldots, h_M) \rightarrow R^M_+$

I. Capacity region under a power-control policy

$$C_f(PC) \equiv \{R : R(S) \leq \mathbb{E}_H \left[ \frac{1}{2} \log \left( 1 + \frac{1}{\sigma^2} \sum_{i \in S} H_i PC_i(H) \right) \right], \ \forall S \subset \{1, \ldots, M\} \}$$

II. Throughput capacity region under power control

**Theorem 2.1:** \(C(\overline{P}) \equiv \bigcup_{PC \in F} C_f(PC)\)

where \(F \equiv \{PC : \mathbb{E}_H[PC(H)] \leq \overline{P}_i \ \forall i\}\)

- **Achievability:** for any $PC$, look the channel as unit power channel with $h_i PC_i(h)$.
  
  $C_f(PC)$ is achievable and $\bigcup_{PC \in F} C_f(PC) \subset C(\overline{P})$

- **Converse:** By Fano’s inequality.
Polymatroid Structure and Greedy Algorithm

I. Polymatroid Structure

Def: \( E = \{1, \ldots, M\} \) and \( f : 2^E \rightarrow R_+ \) be a set function. The polyhedron \( B(f) \equiv \{(x_1, \ldots, x_M) : x(S) \leq f(S) \ \forall S \subset E, x_i \geq 0 \ \forall i\} \) is a polymatroid if the set function \( f \) satisfies

1) \( f(\emptyset) = 0 \) (normalized).
2) \( f(S) \leq f(T) \) if \( S \subset T \) (nondecreasing).
3) \( f(S) + f(T) \geq f(S \cup T) + f(S \cap T) \) (submodular)

II. Greedy Algorithm

Optimization problem
\[
\max \mathbf{1} \cdot \mathbf{x} \quad \text{subject to} \quad \mathbf{x} \in B(f)
\]

- **Initialization:** Set \( x_i = 0 \) for all \( i \). Set \( k = 1 \)

- **Step** \( k \): Increase the value of \( x_{\pi^*(k)} \) until a constraint becomes tight. Goto Step \( k + 1 \)

- After \( M \) steps, the optimal solution is reached.
Polymatroid Structure of Channel Capacity

I. Classical discrete memoryless multiaccess channel

\[ \{ R \in R_+^M : R(S) \leq I[Y; X(S) | X(S^c)] \forall S \subset E \} \text{ is a polymatroid.} \]

II. Memoryless Gaussian multiaccess channel

\[ C_g(h, P) \text{ is a polymatroid.} \]

III. A power control policy \( PC \), \( C_f(PC) \text{ is a polymatroid} \)

Def: A rate allocation policy \( R \) is a mapping: \( h \to R_+^M \)

for each fading state \( h \), \( R_i(h) \) can be interpreted as the rate allocated to user \( i \)

IV. For any power control policy \( PC \)

\[ C_f(PC) = \{ E_h[R(H)] : R \text{ is a rate allocation policy s.t.} \forall h R(h) \in C_g(h, PC(h)) \} \]

Furthermore, \( v(\pi) = E_h[v_h(\pi)] \), for any permutation \( \pi \) on \( E \),

Where \( v(\pi) \leftrightarrow C_f(PC) \),

and \( v_h(\pi) \leftrightarrow C_g(h, PC(h)) \),

for each state \( h \), corresponding to same permutation \( \pi \)
A Lagrangian Characterization of the Capacity Region

Capacity Region: $C_f(PC)$ and $C(\mathbf{P})$

Def: The boundary surface of $C(\mathbf{P})$: is the set of those rates such that no component can be increased with the other components remaining fixed, while remaining in $C(\mathbf{P})$.
Lemma 3.10: I. The boundary surface of $C(\overline{P})$ is the closure of all points $R^*$ such that $R^*$ is a solution to the optimization problem:

$$\max_{R} \mathbf{u} \cdot R \quad \text{subject to} \quad R \in C(\overline{P}) \quad \text{for some} \quad \mathbf{u} \in R^M_+$$

II. For a given $\mathbf{u}$, $R^*$ is a solution to the above problem if and only if there exists a $l \in R^M_+$, rate allocation policy $R(\cdot)$ and power control policy $PC(\cdot)$ such that for every joint fading-state $h$, $(R(h), PC(h))$ is a solution to the optimization problem:

$$\max_{(r,p)} \mathbf{u} \cdot r - l \cdot p \quad \text{subject to} \quad r \in C_g(h, p)$$

and $E_h[R_i(H)] = R^*_i$, $E_h[PC_i(H)] = \overline{P}_i$, $i = 1, ..., M$

where $\overline{P}_i$ is the constraint on the average power of user $i$

| \( \mathbf{u} \) | rate rewards |
| \( l \) | power prices |
| \( h \) | joint fading state |
| \( \overline{P} \) | average power constraints |

$(R(h), PC(h))$
Theorem 3.14: generic problem

\[
\max_{(x,y)} u \cdot x - l \cdot y \quad \text{subject to} \quad x(S) \leq g(y(S))
\]

\(g: \) monotonically increasing concave function

Define the marginal utility functions

\(u_i(z) \equiv u_i g'(z) - l_i, \quad i = 1, \ldots, M\)

\(u^*(z) \equiv \left[ \max_i u_i(z) \right]^+\)

The solution to the problem is:

\[\int_0^\infty u^*(z)dz\]

Time-invariant Gaussian channel

\[
\max \sum_i u_i r_i - \sum_i \frac{l_i}{h_i} q_i \quad \text{subject to} \quad r(S) \leq g(q(S))
\]

\(g(z) \equiv \frac{1}{2} \log(1 + \frac{z}{\sigma^2})\)

\(u_i(z) \equiv \frac{u_i}{2(\sigma^2 + z)} \frac{l_i}{h_i}\)

\(u^*(z) \equiv \left[ \max_i u_i(z) \right]^+\)

\[\int_0^\infty u^*(z)dz\]
A three-user example illustrating the greedy power allocation

With probability 1, the optimal power and rate allocation is unique and is explicitly given by

\[ R_i^*(h) = \int_{A_i} \frac{1}{2(\sigma^2 + z)} dz \]

where

\[ A_i = \{ z \in [0, \infty) : u_i(z) > u_j(z) \ \forall \ j \neq i \ \text{and} \ u_i(z) > 0 \} \]
Boundary of the Capacity Region $C(\overline{\mathbf{P}})$

I. Lemma 3.15: (Uniqueness) for $\mathbf{u} \in R_+^M$, there is a unique $\mathbf{R}^*$ on the boundary which maximizes $\mathbf{u} \cdot \mathbf{R}$, and there is a unique Lagrangian power price $\mathbf{l}$ such that the optimal power allocation satisfies the average power constraints.

II. Theorem 3.16: For independent users' fading processes, the boundary of $C(\overline{\mathbf{P}})$ is the closure of the parametrically defined surface

$$\{\mathbf{R}^*(\mathbf{u}) : \mathbf{u} \in R_+^M, \sum_i u_i = 1\}$$

where

$$R_i^*(\mathbf{u}) = \int_0^\infty \frac{1}{2(\sigma^2 + z)} \{\int_0^\infty \prod_{k \neq i} F_k\left(\frac{2l_k h(\sigma^2 + z)}{2l_i (\sigma^2 + z) + (u_k - u_i)h}\right) f_i(h) dh\} dz$$

and where the vector $\mathbf{l}$ is the unique solution of the equations

$$\int_0^\infty \{\int_0^\infty \frac{1}{h} \prod_{k \neq i} F_k\left(\frac{2l_k h(\sigma^2 + z)}{2l_i (\sigma^2 + z) + (u_k - u_i)h}\right) f_i(h) dh\} dz = \overline{P}_i$$

$F_i$ and $f_i$ are the cdf and pdf of the stationary fading distribution of user $i$.
Cases for throughput capacity and power allocation in fading channels

I. Single-User Channel

\[ R^* = \int_0^\infty \frac{1}{2(\sigma^2 + z)} \left\{ \int_0^\infty \frac{h}{\sigma^2} \cdot f(h) dh \right\} dz = \int_0^\infty \frac{1}{2} \log(1 + \frac{h}{\sigma^2} \left( \frac{u}{2l} - \frac{\sigma^2}{h} \right)^+) f(h) dh \]

\[ \frac{u}{2l} \] satisfies the power constraint

\[ \int_0^\infty \left( \frac{u}{2l} - \frac{\sigma^2}{h} \right)^+ f(h) dh = P \]

Note:

• Time Water-Filling Solution: Power allocation over a set of parallel single-user channels, one for each fading level \( h \)

• More power is used when the channel is good and little or even no power when it is bad.
II. Maximum Sum-Rate Point \((u_1 = \ldots = u_M = 1)\)

Utility Functions:

\[ u_i(z) = \frac{1}{2(\sigma^2 + z)} - \frac{l_i}{h_i} \]

Power Control Strategy:

\[ PC_i^*(h, l) = \begin{cases} 
    \left(\frac{1}{2l_i} - \frac{\sigma^2}{h_i}\right)^+, & \text{if } h_i > \frac{l_i}{l_j} h_j \text{ for all } j \\
    0, & \text{else.}
\end{cases} \]

Optimal Rates:

\[ R_i^* = \int_0^\infty \frac{1}{2} \log(1 + \frac{h}{\sigma^2} \left(\frac{1}{2\lambda_i} - \frac{\sigma^2}{h}\right)^+) \times \prod_{k \neq i} F_k\left(\frac{l_k h}{l_i}\right) f(h)dh \]

Power Prices Constant:

\[ \int_0^\infty \left(\frac{1}{2l_i} - \frac{\sigma^2}{h}\right)^+ \prod_{k \neq i} F_k\left(\frac{l_k h}{l_i}\right) f(h)dh = P_i, \]

Note:

- Here the optimal rates are in the sense of sum of all users’ rates.
- The optimal (TDMA) strategy allows at most one user to transmit at any given fading state, this lucky user has the best channel and largest available power.
- As for the two-user example, the boundary point corresponding to \( u_1 = u_2 \) is the corner point of a rectangular \( C_f(PC) \).
III. Multiple Classes of Users

If the fading processes of the users have very different statistics, in order to equalize users’ rates, unequal rate awards can be assigned to users.

Two classes users example:
Class 1: *far* users at the cell boundary, all assigned rate award $u_1$
Class 2: *near* users close to the base station, all assigned rate award $u_2$

$u_1 > u_2$

Note:
- At each fading state, only the strongest user in each class transmits.
- The two strongest users are decoded by successive cancellation, with nearby user decoded first.
Appendix:

The channel capacity for the receiver tracking the channel only.

\[ C = \max_{p(x)} I(X; Y, H) = \max_{p(x)} [I(X; H) + I(X; Y | H)] \]

( \( I(X; H) = 0 \) --- the fading variable and the channel input are independent )

\[ = \max_{p(x)} E[I(X; Y | H = h)] \]

\[ \{(R_1, ..., R_M) : R(S) \leq E_H \left[ \frac{1}{2} \log(1 + \frac{\sum_{i \in S} H_i P_i}{\sigma^2}) \right], \forall S \subset \{1, ..., M\}\} \]
Appendix:

Capacity region under a power-control policy

\[ C_f(PC) = \{ R : R(S) \leq \mathbb{E}_H \left[ \frac{1}{2} \log(1 + \frac{1}{\sigma^2} \sum_{i \in S} H_i PC_i(H)) \right], \quad \forall S \subset \{1,...,M\} \} \]
Appendix:

\[ B(f) \equiv \{(x_1, \ldots, x_M) : x(S) \leq f(S) \ \forall S \subset E, x_i \geq 0 \ \forall i\} \text{ is a polymatroid} \]

\( \pi \) is a permutation on the set \( E \), define the vector \( v(\pi) \in R^M \) by

\[
\begin{align*}
v_{\pi(1)}(\pi) &= f(\pi(1)) \\
v_{\pi(i)}(\pi) &= f(\{\pi(1), \ldots, \pi(i)\}) - f(\{\pi(1), \ldots, \pi(i-1)\}) & i = 2, \ldots, M.
\end{align*}
\]

Then \( v(\pi) \) is a vertex of \( B(f) \) for every permutation \( \pi \).

Conversely, suppose \( f \) is a set function and \( B(f) \) is the polyhedron defined as above. Then if \( v(\pi) \in B(f) \) for every permutation \( \pi \), then \( B(f) \) is a polymatroid.

Consider the polyhedron \( \{R \in R_+^M : R(S) \leq I[Y; X(S) | X(S^c)] \forall S \subset E\} \)

\[ f(S) = I[Y; X(S) | X(S^c)] \]

\( \pi \) is a permutation on the set \( E \), define the vector \( R(\pi) \in R^M \) by

\[
\begin{align*}
R_{\pi(1)}(\pi) &= I[Y; X_{\pi(1)} | X(\{\pi(2), \ldots, \pi(M)\})] \\
R_{\pi(i)}(\pi) &= I[Y; X_{\pi(i)} | X(\{\pi(i+1), \ldots, \pi(M)\})], & i = 1, \ldots, M - 1 \quad \text{(Chain rule)} \\
R_{\pi(M)}(\pi) &= I[Y; X_{\pi(M)}]
\end{align*}
\]

Obviously, \( R(\pi) \) lies in \( R(S) \), so the polyhedron region is a polymatroid.
Appendix:

A Lagrangian multiplier:

Optimization problem: \[
\max_R u \cdot R \quad \text{subject to} \quad R \in C(\bar{P})
\]
\[
\Downarrow
\]
\[
\max_{R \in C(P)} u \cdot R \quad \text{subject to} \quad P = \bar{P}
\]

Lagrangian multipliers

\[
\Downarrow
\]

there exists an \( \mathbf{l} \), \[
\max_{R \in C(P)} u \cdot R - \mathbf{l} \cdot (P - \bar{P})
\]
\[
\Downarrow
\]

there exists an \( \mathbf{l} \), \[
\max_{R \in C(P)} u \cdot R - \mathbf{l} \cdot P
\]
Appendix:

The classic water-filling solution for the single-user case

For each fading state \( h \), the optimization problem:

\[
\max_{r,q} [r - \frac{l}{h} q] \quad \text{subject to} \quad r \leq \frac{1}{2} \log(1 + \frac{q}{\sigma^2})
\]

\[
\downarrow
\]

\[
\max_q \left[ \frac{1}{2} \log(1 + \frac{q}{\sigma^2}) - \frac{l}{h} q \right]
\]

\[
\downarrow
\]

\[
\frac{1}{2} \log(1 + \frac{q}{\sigma^2}) = \int_0^q \frac{1}{2\sigma^2 + z} \, dz
\]

\[
\max_q \int_0^q \left[ \frac{1}{2(\sigma^2 + z)} - \frac{l}{h} \right] dz
\]

received power

... 

marginal utility function

\[
u(z) \equiv \frac{1}{2(\sigma^2 + z)} - \frac{l}{h}
\]

Optimization problem turns out to be:

Adding more virtual users until \( u(z) \cdot dz < 0 \)