Improving Control Performance across AWGN Channels using a Relay Node

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Abstract

Consider an unstable linear time invariant system in which the sensor transmits information to a controller across an additive white Gaussian noise channel. The designer can optionally utilize a relay node to assist the controller; however the total transmission power consumed by the sensor and the relay node is constant. We consider two topologies: (i) a Gaussian relay channel, and (ii) a cascade of two Gaussian point to point channels. We propose coding schemes and present sufficient conditions for the stabilizability of the plant through such schemes. The analysis suggests that it is useful to provide a relay node, even if the total transmission power remains the same.

1 Introduction

Consider an unstable linear time invariant (LTI) discrete time system being stabilized across an additive white Gaussian channel, when a relay is present. If the relay is not present, stabilizability conditions in terms of the signal to noise ratio (SNR) of the channel and the unstable eigenvalues of the system have been derived (e.g., [1,2,5,8]) usually using coding schemes such as the Schalkwijk-Kailath (SK) scheme [9] that achieve capacity for a Gaussian channel with feedback. [3] has also addressed the problem of stabilizing two scalar linear time invariant systems over noisy multiple-access and broadcast communication channels. In information theory, it is known that relays can increase the capacity of a Gaussian channel, even if the total transmission power (consumed together by the encoder and the relay) remains the same (e.g., [4]). However, mean square stabilizability of an unstable process across a communication channel requires that the error about the state value at the controller decays doubly exponentially [7]. It is not clear if relays can be used to expand the stabilizability region. There does not seem to be a counterpart of the SK scheme for relay channels.

In this paper, we develop sufficient conditions for stabilizing a discrete-time LTI plant in the mean square sense by a remotely placed controller over a Additive White Gaussian (AWGN) relay channel. We consider and propose coding schemes for two scenarios: (i) one in which the controller has a direct path from the sensor (the relay channel), and (ii) another in which the controller does not have a direct path from the sensor (the cascade channel). Interestingly, our analysis suggests that the stabilizability region can be increased by using a relay node even if the total transmission power remains the same.

The paper is organized as follows. The problem is setup formally in Section 2. The encoder and decoder design at the sensor, relay node and controller are presented and analyzed in Section 3. The stabilizability conditions are numerically illustrated in Section 4.

Notation: We denote the set of real numbers by $\mathbb{R}$ and that of positive integers by $\mathbb{Z}_+$. Denote the i-th basis vector by e_i. Thus, $e_i \in \mathbb{R}^m$ has all elements 0, except for the i-th one which is unity. By $\log(x)$ we mean logarithm to the base 2. Denote the m eigenvalues of matrix M by $\lambda_j(M)$, $j = 1, \cdots, m$.

2 Problem Setup

Consider Figs. 1 and 2 with the following asumptions:
Process: Consider an open loop unstable linear time invariant process evolving as

$$S(k+1) = AS(k) + BU(k), \tag{1}$$

where $S(k) \in \mathbb{R}^m$ is the state and $U(k) \in \mathbb{R}$ is the control value. The initial condition $S(0)$ is assumed to be a random variable with a finite covariance $\sigma^2_{S(0)}$ and an arbitrary probability distribution. For simplicity, we

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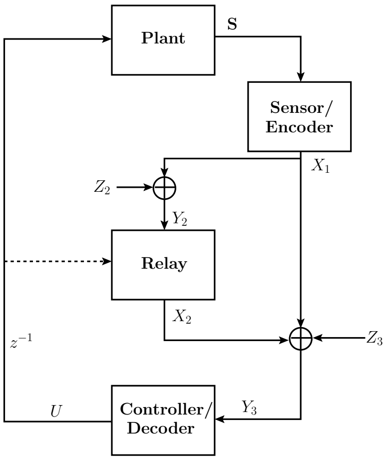

Fig. 1. Block diagram for an unstable plant being controlled by a controller across an AWGN relay channel

assume that the process state is observed. We assume that the pair (A, B) is controllable.

Communication channel: In this work, we model a wireless link as an additive white Gaussian noise channel:

$$Y(k) = gX(k) + Z(k), \tag{2}$$

where $X(k)$ and $Y(k)$ are the input and output respectively and $Z(k)$ is white Gaussian noise with mean 0 and covariance $\Sigma_Z > 0$. The factor g is an attenuation factor that depends on the distance d between the transmitter and receiver. A typical model is $g = bd^{-\eta/2}$, where η is the path loss exponent (typically $2 \leq \eta \leq 4$ for wireless communication), and b is a constant. This model usually holds only for distances d, for which $bd^{-\eta/2} \ll 1$. However, since the constant terms b and η do not affect the analysis that follows in terms of relative performance, we have assumed the same path loss exponent model in order to simplify the notation.

Sensor to controller communication: In the baseline case, the sensor transmits to the controller that is located a distance d away across a channel of the form (2). If a relay is present, let it be placed at a distance d_1 from the sensor and d_2 from the controller. The sensor transmits a quantity $X_1(k) \in \mathbb{R}$ which is a causal function $X_1(k) = f\left(S(0), \ldots, S(k-1)\right)$ of the information to which it has access, while satisfying a power constraint

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}[X_1^2(k)] \leq P_1. \tag{3}$$

We consider two cases. In the first case (the relay channel) shown in Figure 1, a noisy version of $X_1(k)$ is received at both the relay and the controller. In the second case (the cascade channel) shown in Figure 2, $X_1(k)$ is not received at the controller. We denote the quantity received at time k at the relay from the sensor

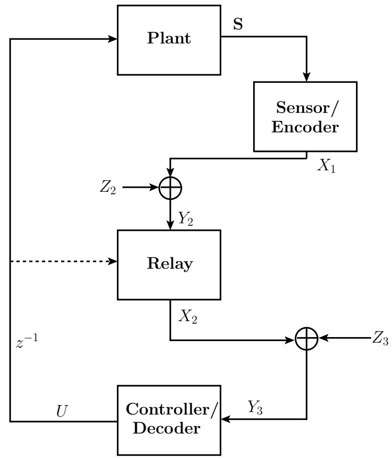

Fig. 2. Block diagram for an unstable plant being controlled by a controller across an AWGN cascade channel

by $Y_2(k)$ and the corresponding noise by $Z_2(k)$, i.e., $Y_2(k) = g_{12}X_1(k) + Z_2(k)$, where $g_{12} = bd_1^{-\eta/2}$. The relay node transmits a quantity $X_2(k)$ that is a causal function of the information to which it has access. Moreover the relay satisfies a power constraint

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}[X_2^2(k)] \leq P_2. \tag{4}$$

In the relay channel case, the controller receives $Y_3(k) = g_{13}X_1(k) + g_{23}X_2(k) + Z_3(k)$, where $g_{13} = bd^{-\eta/2}$ and $g_{23} = bd_2^{-\eta/2}$. In the cascade channel case, $Y_3(k) = g_{23}X_2(k) + Z_3(k)$. We assume that the noises $Z_2(.)$ and $Z_3(.)$ are mutually independent and white with mean 0 and variances σ_2^2 and σ_3^2 respectively. Note that if $P_1 + P_2 = P$, then the relay and the sensor node share the power of the baseline case. In such a case, it is not clear if the addition of a relay node will help.

Controller to actuator communication: The controller calculates a control input according to a causal function $U(k) = h(U(0), \cdots, U(k-1), Y_3(0), \cdots, Y_3(k))$ and transmits it to the actuator.

Constraint C_1: The control action must satisfy a controller cost constraint, $\sum_{k=0}^{\infty} \mathbb{E}[U^2(k)] < \infty$.

We assume that the controller is not power limited; thus, the actuator receives $U(k)$ without corruption. Due to the broadcast nature of the communication, the sensor and the relay also receive the control $U(k)$.

Problem statement: We are interested in designing the functions $X_1(.)$ and $X_2(.)$ (that we refer to as the design of the coding scheme), and of $U(.)$ (that we refer to as the design of the controller), such that the process (1) is mean squared stabilized while the constraints C_1, (3) and (4) are satisfied. In particular, can the stabilizability region of the process be enhanced using a relay node even if the total transmission power used by the sensor and the relay node stays constant?

3 Main Results

Baseline Case: The stabilizability conditions across a point-to-point AWGN channel are as follows [1, 5, 8].

Theorem 1 *Consider the problem formulation presented in Section 2 with the process given by (1) without a relay node. The process (1) is mean square stabilized if*

$$\sum_{i=1}^{m} \max\{0, \log(\lambda_i(A))\} < \frac{1}{2}\log\left(1 + \frac{g_{1,3}^2 P}{\sigma_3^2}\right). \quad (5)$$

Case with Relay: The solution in the presence of a relay node has two components. We develop an encoding scheme that ensures that at every time k, the controller can calculate an estimate $\hat{S}_3(k)$ of the state value $S(0)$ such that the error $\epsilon_3(k) = \hat{S}_3(k) - S(0)$ has a covariance that decreases geometrically in k. We then show that using such a $\hat{S}_3(k)$, the controller can calculate a control input that stabilizes the process in the mean squared sense. We begin by proving the second part.

Theorem 2 *(Controller Design) Consider the problem formulation stated in Section 2. Assume that*

$$\mathbb{E}[\epsilon_3(k)] = 0, \quad (6)$$

$$\lim_{k\to\infty} A^k \mathbb{E}[\epsilon_3(k)\epsilon_3^T(k)](A^T)^k = 0. \quad (7)$$

Then, the process (1) can be mean squared stabilized by a suitable choice of the controller.

PROOF. Let K be such that the closed loop matrix $A + BK$ is Schur-stable. Such a K exists since (A, B) is controllable. Consider the controller $U(k) = K\hat{S}(k)$, where

$$\hat{S}(k) = A^k \hat{S}_3(k) + \sum_{j=1}^{k} A^{k-j} BU(j-1). \quad (8)$$

With this controller, the process (1) evolves as

$$S(k+1) = (A + BK)S(k) + BK\delta(k), \quad (9)$$

where $\delta(k) = S(k) - \hat{S}(k) = -A^k\epsilon_3(k)$. Since $S(0)$ is zero mean, (6) implies that $\mathbb{E}[\delta(k)] = 0$ and in turn $\mathbb{E}[S(k)] = 0$ at every k. Moreover, if (7) is satisfied, $\mathbb{E}[\delta(k)\delta^T(k)] \to 0$ as $k \to \infty$. Thus, since $A + BK$ is stable, (9) yields that $\lim_{k\to\infty} \mathbb{E}[S(k)S^T(k)] = 0$. $\square$

Note that even though a certainty equivalence controller is used, the controller does not generate its estimate using a Kalman filter. This result also provides the controller design. We now provide a coding scheme that ensures that the relations (6) and (7) are satisfied.

Scalar LTI plant: For pedagogical ease, we begin by describing the scheme for the case $m = 1$. For this special case, denote the matrix A in (1) by a (with $a > 1$) and assume the matrix B is unity. In our coding scheme, the sensor and relay transmit estimation errors with respect to the estimate at the controller, $\hat{S}_3(k)$. Specifically, the relay and controller calculate linear mean squared estimates (MMSE) $\hat{S}_2(k)$ and $\hat{S}_3(k)$ respectively, of the initial condition $S(0)$. Define $\epsilon_2(k) := \hat{S}_2(k) - S(0)$ and $\epsilon_{3,2}(k) := \hat{S}_3(k) - \hat{S}_2(k)$. Let $\alpha_i(k)$ represent the covariance of $\epsilon_i(k), i \in \{\{2\}, \{3\}, \{3, 2\}\}$. At every time k, the sensor can calculate $\epsilon_3(k)$, while the relay can calculate $\epsilon_{3,2}(k)$. Since the relay and controller calculate linear MMSE estimates, the proposed coding scheme will ensure that the quantities $\epsilon_3(k)$ and $\epsilon_{3,2}(k)$ are jointly Gaussian with zero means, covariances $\alpha_3(k)$ and $\alpha_{3,2}(k)$ (respectively), and correlation coefficient $\rho(k)$. The quantities $\alpha_2(k)$, $\alpha_3(k), \alpha_{3,2}(k)$ and $\rho(k)$ are deterministic and can be calculated prior to transmission by all the nodes. In particular, $\alpha_{3,2}(0) = \alpha_3(0) + \alpha_2(0)$ and $\rho(0) = \frac{\alpha_3(0)}{\sqrt{(\alpha_3(0)+\alpha_2(0))\alpha_3(0)}}$. Recursive expressions to calculate these quantities are provided later.

<u>Coding Scheme</u>: We now proceed to explain the coding scheme used at the sensor and the relay. We concentrate on the relay channel while noting that the coding scheme and its analysis carry over to the cascade channel if we set the channel gain $g_{1,3} = 0$. The scheme works as follows.

Initialization: At time step $k = 0$,

- The sensor observes $S(0)$ and transmits the input $X_1(0)$ given by

$$X_1(0) = \sqrt{\frac{P_1}{\sigma_{S(0)}^2}} S(0). \quad (10)$$

- The relay node transmits nothing. It receives $Y_2(0) = g_{1,2}X_1(0) + Z_2(0)$, and calculates an estimate of $S(0)$ by scaling: $\hat{S}_2(0) = \frac{1}{g_{1,2}}\sqrt{\frac{\sigma_{S(0)}^2}{P_1}} Y_2(0)$. The estimation error $\epsilon_2(0)$ zero-mean Gaussian with variance

$$\alpha_2(0) = \frac{\sigma_{S(0)}^2 \sigma_2^2}{g_{1,2}^2 P_1}. \quad (11)$$

- The controller node 3 receives $Y_3(0) = g_{1,3}X_1(0) + Z_3(0)$ and calculates an estimate of $S(0)$ as $\hat{S}_3(0) = \frac{1}{g_{1,3}}\sqrt{\frac{\sigma_{S(0)}^2}{P_1}} Y_3(0)$. The estimation error $\epsilon_3(0)$ is again zero-mean Gaussian with variance

$$\alpha_3(0) = \frac{\sigma_{S(0)}^2 \sigma_3^2}{g_{1,3}^2 P_1}. \quad (12)$$

The controller calculates $\hat{S}(0)$ from (8) and transmits both the control $U(0) = K\hat{S}(0)$ and the estimate $\hat{S}_3(0)$.

Update: At every time step $k \geq 1$,

• Using the message transmitted by the controller at time $k-1$, the sensor and the relay have access to $\hat{S}_3(k-1)$. The sensor calculates $\epsilon_3(k-1)$, while the relay calculates the quantity $\epsilon_{3,2}(k-1)$.

• The sensor transmits $X_1(k) = \sqrt{\frac{P_1}{\alpha_3(k-1)}}\epsilon_3(k-1)$.

• The relay transmits $X_2(k) = \sqrt{\frac{P_2}{\alpha_{3,2}(k-1)}}\epsilon_{3,2}(k-1)$. It also updates its estimate as follows. It first calculates

$$\begin{aligned}
&Y_2'(k)\\
&= Y_2(k) - g_{1,2}\sqrt{\frac{P_1}{\alpha_3(k-1)}}\left(\hat{S}_3(k-1) - \hat{S}_2(k-1)\right)\\
&= g_{1,2}\sqrt{\frac{P_1}{\alpha_3(k-1)}}\left(\hat{S}_2(k-1) - S(0)\right) + Z_2(k). \qquad (13)
\end{aligned}$$

Using this quantity, the relay calculates the linear minimum mean squared error estimate of $S(0)$ given $Y_2'(k)$ and $\hat{S}_2(k-1)$ as

$$\hat{S}_2(k) = \hat{S}_2(k-1) - \beta_2(k)Y_2'(k) \qquad (14)$$

$$\beta_2(k) = \frac{\mathbb{E}[Y_2'(k)\epsilon_2(k-1)]}{\mathbb{E}[{Y_2'}^2(k)]}. \qquad (15)$$

Note that $\beta_2(k)$ can be calculated off-line. We provide an expression for calculation of $\beta_2(k)$ later in (19).

• At time $k \geq 1$, the controller calculates the linear MMSE of $S(0)$ given $Y_3(k)$ and $\hat{S}_3(k-1)$ as

$$\hat{S}_3(k) = \hat{S}_3(k-1) - \beta_3(k)Y_3(k) \qquad (16)$$

$$\beta_3(k) = \frac{\mathbb{E}[Y_3(k)\epsilon_3(k-1)]}{\mathbb{E}[{Y_3}^2(k)]}. \qquad (17)$$

We provide an expression for calculation of $\beta_3(k)$ in (22). Finally, the controller then calculates the control input $U(k) = K\hat{S}(k)$ using equation (8) and transmits it.

For future reference, we denote the coding scheme by $\mathcal{S}(S(0), \hat{S}_3(k))$, where $S(0)$ is the initial condition, and $\hat{S}_3(k)$ is the estimate of $S(0)$ at time k at the controller.

<u>Calculation of the Variances:</u> We now provide recursive expressions for the calculation of $\alpha_2(k)$, $\alpha_3(k)$, $\alpha_{3,2}(k)$, $\rho(k)$, $\beta_2(k)$ and $\beta_3(k)$ as used in the coding scheme presented above. The following recursions do not depend on the data, and can be executed by any node.

Variance $\alpha_2(k)$ of the error at the relay node: Since $\alpha_2(k) = \mathbb{E}[\epsilon_2^2(k)]$, equations (14), and (15) yield $\alpha_2(k) = \alpha_2(k-1) - \frac{\mathbb{E}^2[Y_2'(k)\epsilon_2(k-1)]}{\mathbb{E}[{Y_2'}^2(k)]}$ with the initial condition in equation (11). Simplifying, we can write $\alpha_2(k) = \alpha_2(k-1)r(k-1)$, where

$$r(k-1) = \left(\frac{\sigma_2^2}{g_{1,2}^2\frac{P_1}{\alpha_3(k-1)}\alpha_2(k-1) + \sigma_2^2}\right). \qquad (18)$$

It can also be verified that $\beta_2(k)$ can be calculated as

$$\beta_2(k) = \frac{g_{1,2}\sqrt{\frac{P_1}{\alpha_3(k-1)}}\alpha_2(k-1)}{g_{1,2}^2\frac{P_1}{\alpha_3(k-1)}\alpha_2(k-1) + \sigma_2^2}. \qquad (19)$$

Variance $\alpha_3(k)$ of the error at the controller node: In a similar manner, the variance $\alpha_3(k) = \mathbb{E}[\epsilon_3^2(k)]$ can be calculated from equations (16), and (17) to be

$$\alpha_3(k) = \alpha_3(k-1)q(k-1), \qquad (20)$$

with

$$q(k-1) = \frac{g_{2,3}^2P_2\left(1-\rho^2(k-1)\right)+\sigma_3^2}{g_{1,3}^2P_1+g_{2,3}^2P_2+2g_{1,3}g_{2,3}\sqrt{P_1P_2}\rho(k-1)+\sigma_3^2} \qquad (21)$$

and the initial condition in equation (12). It can be verified that $\beta_3(k)$ can be calculated as

$$\beta_3(k) = \frac{\sqrt{\alpha_3(k-1)}\left(g_{1,3}\sqrt{P_1}+g_{2,3}\sqrt{P_2}\rho(k-1)\right)}{g_{1,3}^2P_1+g_{2,3}^2P_2+2g_{1,3}g_{2,3}\sqrt{P_1P_2}\rho(k-1)+\sigma_3^2}. \qquad (22)$$

Variance $\alpha_{3,2}(k)$ and the correlation coefficient $\rho(k)$: We express $\mathbb{E}\left[\epsilon_{3,2}(k)\epsilon_2(k)\right] = \gamma(k) - \alpha_2(k)$ where $\gamma(k) := \mathbb{E}\left[\epsilon_3(k)\epsilon_2(k)\right]$. Now, equations (14) and (16) yield

$$\begin{aligned}
\gamma(k) &= \left(\gamma(k-1) - \beta_2(k)\mathbb{E}[\epsilon_3(k-1)Y_2'(k)]\right)\\
&\quad - \left(\beta_3(k)\mathbb{E}[\epsilon_2(k-1)Y_3(k)] - \beta_3(k)\beta_2(k)\mathbb{E}[Y_2'(k)Y_3(k)]\right),
\end{aligned}$$

which yields a recursive relation for $\gamma(k)$, with the initial value $\gamma(0) = 0$. Finally, given $\alpha_2(k)$, $\alpha_3(k)$, and $\gamma(k)$, we can calculate $\alpha_{3,2}(k)$ and $\rho(k)$ using $\alpha_{3,2}(k) = \alpha_3(k) + \alpha_2(k) - 2\gamma(k)$, and $\rho(k) = \frac{\alpha_3(k)-\gamma(k)}{\sqrt{\alpha_3(k)\alpha_{3,2}(k)}}$.

<u>Stability Analysis:</u> We now present the stability conditions if the coding scheme described above is used.

Theorem 3 *Consider the problem formulation presented in Section 2 for a scalar LTI plant with the process given by (1) and the coding scheme and controller presented above. The process (1) is mean square stabilized over the AWGN relay channel if*

$$\log(a) < \liminf_{k\to\infty} \frac{1}{2}\log\frac{1}{q(k-1)}. \qquad (23)$$

PROOF. Since $\mathbb{E}[\epsilon_3(0)] = \mathbb{E}[\epsilon_{3,2}(0)] = \mathbb{E}[\epsilon_2(0)] = 0$ and all updates in the coding scheme are linear, it is straight-forward to see that equation (6) is satisfied. Now, (20) yields $a^{2k}\alpha_3(k) = a^{2k}q(k-1)\alpha_3(k-1)$. The ratio test then yields that a sufficient condition for equation (7) to be satisfied is that the condition in (23) be satisfied. Since the coding scheme in Section 3 uses the controller used in the proof of Theorem 2, the process (1) is mean square stabilized. $\square$

Corollary 4 *The right hand side of (23) is finite.*

PROOF. From (21), we see that $\rho(k-1) = 1$ minimizes the numerator as well as maximizes the denominator. Thus, $1/q(k-1)$ is maximized by choosing $\rho(k-1) = 1$, or in other words,

$$\frac{1}{q(k-1)} \leq \frac{g_{1,3}^2 P_1 + g_{2,3}^2 P_2 + 2g_{1,3}g_{2,3}\sqrt{P_1 P_2} + \sigma_3^2}{\sigma_3^2},$$

from which the result follows. $\square$

The result for the cascade channel can be derived from the above results by substituting $g_{1,3} = 0$, which ensures that there is no path from the sensor to the controller.

Theorem 5 *The LTI system in (1) can be mean square stabilized over the cascade channel if (23) holds with* $g_{1,3} = 0$.

Vector LTI plant: We now consider the process in (1) to have an arbitrary dimension m. Without loss of generality, we assume that the matrix A is in the modal form and can be expressed as $A = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix}$, where $A_s \in \mathbb{R}^{(m-n)\times(m-n)}$ and $A_u^{-1} \in \mathbb{R}^{n\times n}$ are Schur stable. Note that $0 \leq n \leq m$, and we assume that an empty A_s (resp. A_u) corresponds to $n = m$ (resp. $n = 0$).

Coding Scheme: The basic approach for a vector process is to transmit the last n elements of the initial state $S(0)$ to the controller. To achieve this aim, n coding schemes of the type proposed for scalar LTI plants are used in parallel (one for each element of $S(0)$). More formally, for each $j = 0, 1, \cdots, n-1$, at the sequence of times $kn + j$ ($k \in \mathbb{Z}_+$), the sensor, relay, and controller implement the coding scheme $\mathcal{S}(S^j(0), S_3^j(k))$, with $S^j(k) = e_j^T S(0)$ and $S_3^j(k) = e_j^T S_3(k)$. The controller calculates the control input as follows. It maintains an estimate $\hat{S}_3(k)$ of the initial state $S(0)$. At each time k, such that $j = k \mod n$, it performs the following actions:

- Update $\hat{S}_3(k)$ as $\hat{S}_3(k) = \hat{S}_3(k-1) - e_j^T \hat{S}_3(k-1)e_j + S_3^j(k)e_j$, with the initial condition $\hat{S}_3(-1) = 0$.
- Calculate $\hat{S}(k)$ using the relation (8).
- Transmit control $U(k) = K\hat{S}(k)$ and estimate $S_3^j(k)$.

Stability Analysis: We have the following result.

Theorem 6 *Consider the problem formulation presented in Section 2 with the coding scheme and controller presented above. The process (1) with n unstable states is mean square stabilized over the AWGN relay channel if the following condition is satisfied:*

$$\sum_{i=1}^{m} \max\{0, \log|\lambda_i(A)|\} < \liminf_{k\to\infty} \frac{1}{2}\log\frac{1}{q(k)}, \qquad (24)$$

where $q(k-1)$ is calculated as in equation (21).

PROOF. Since, A_s is stable, we do not need to update the first $m-n$ components of the error vector ϵ_3, which remain constant at 0. The other n components of ϵ_3 are updated every n time steps. Since $\epsilon_3(0) = 0$ and all updates in the coding scheme are linear, it is straightforward to see that equation (6) is satisfied. Since A_s is Schur stable, A_s^k approaches 0 as $k \to \infty$. Using the result from Theorem 3, we can see that $\lambda_j^{2k}\mathbb{E}\left[(\epsilon_3^j(k))^2\right] \to 0$, where $m-n+1 \leq j \leq m$ and $\epsilon_3^j(k) = e_j^T\epsilon_3(k)$, if the condition $\log|\lambda_j(A)| < \liminf_{k\to\infty}\frac{1}{2n}\log\frac{1}{q(k)}$, is satisfied where $q(k-1)$ is calculated as in equation (21). Note that since the diagonal elements in (7) approach 0 as $k \to \infty$, using Cauchy-Schwarz inequality, it follows that the non-diagonal elements in (7) also approach 0. The theorem follows by noting that the above condition needs to be satisfied for $\forall 1 \leq i \leq m$. $\square$

The only parameter that depends on the choice of design parameters such as the transmission power is $q(k)$. Consequently, we can rewrite the above result as follows.

Corollary 7 *Let the assumptions of Theorem 6 hold. The minimum powers at the sensor and the relay nodes that are needed for stabilizing the process (1) with n unstable states in the mean squared sense are given by*

$$\liminf_{k\to\infty} \log\frac{1}{q(k)} > 2\sum_{i=1}^{m}\max\{0, \log|\lambda_i(A)|\}, \qquad (25)$$

where $q(k-1)$ is calculated as in equation (21).

Constraint C_1:

Proposition 8 *If $\hat{S}_3(k)$ is a linear MMSE estimate of $S(0)$, and the process (1) is stabilized in the mean squared sense, then the controller proposed in Theorem 2 satisfies the constraint C_1.*

PROOF. First note that if $\hat{S}_3(k)$ is an MMSE estimate of $S(0)$, then $\hat{S}(k)$ in (8) is an MMSE estimate of $S(k)$. This implies that $\mathbb{E}[S(k)S^T(k)] = \mathbb{E}[\hat{S}(k)\hat{S}^T(k)] + \mathbb{E}[\delta(k)\delta^T(k)]$. Thus,

$$\sum_{k=0}^{\infty} \mathbb{E}[U^T(k)U(k)] = \sum_{k=0}^{\infty} \mathbb{E}[tr(\hat{S}(k)\hat{S}^T(k)K^T K)],$$

which is finite if $\sum_{k=0}^{\infty} \mathbb{E}[tr(\hat{S}(k)\hat{S}^T(k))]$ is finite. Now, $\sum_{k=0}^{\infty} \mathbb{E}[tr(\hat{S}(k)\hat{S}^T(k))] = \sum_{k=0}^{\infty} \mathbb{E}[tr(S(k)S^T(k))] + \sum_{k=0}^{\infty} \mathbb{E}[tr(\delta(k)\delta^T(k))]$. If the process (1) is mean squared stabilized, the first summation is finite. The second summation can be written as

$$\begin{aligned}\sum_{k=0}^{\infty} \mathbb{E}[tr(\delta(k)\delta^T(k))] &= \sum_{k=0}^{\infty}\sum_{j=1}^{m} \lambda_j^{2k} \mathbb{E}[(\epsilon_3^j(k))^2] \\ &\leq \sum_{j=1}^{m}\sum_{k=0}^{\infty} \alpha_3(0) n \prod_{i=0}^{k-1} \lambda_j^{2n} q(i),\end{aligned}$$

where the last step follows from the fact that a particular estimation error is updated every n time steps. If the condition in (24) is satisfied, the above term is finite. Thus, the constraint C_1 is satisfied. □

4 Numerical Results

Consider a scalar process of the form (1) with the controller placed at a (spatial) distance of 2 units from the sensor. The relay is placed at the midpoint of the line joining the sensor and controller. The system parameters are considered to be $\sigma_{S(0)}^2 = \sigma_2^2 = \sigma_3^2 = 1$ and $\eta = 2$. All the logarithms are taken to the base 2. We plot the limit in (23) as a function of the power P_1 and distance of the relay from the plant in Figure 3. The total power is fixed at $P_1 + P_2 = 1$. As shown, for any given position of the relay, we can always find a power distribution for which the stability region is enhanced.

5 Conclusions

In this paper, we derived sufficient conditions for stabilizability of a linear time invariant open loop unstable plant in the mean squared sense over a AWGN relay channel and a cascade of two point-to-point AWGN channels. We proposed a coding scheme to ensure a sufficiently fast rate of convergence of the estimate of the initial state at the controller to the correct value. The scheme can also be interpreted as a distributed version of the Schalkwijk-Kailath coding scheme for point-to-point channels with both the sensor and the relay transmitting innovations with respect to the estimate at the controller at every step. We analyzed the stability region of the closed loop system with the proposed scheme under average transmission power constraints for both the sensor and the relay node. Interestingly, the stability region may be increased by using a relay node even if the total transmission power remains the same.

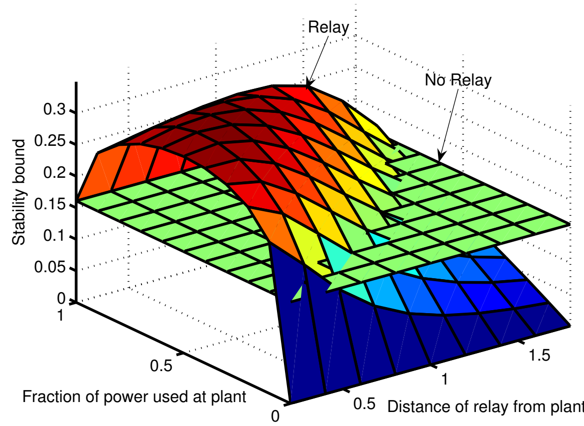

Fig. 3. Sufficient condition bound as a function of fraction of power used at the plant and distance of relay from plant.

References

[1] J. H. Braslavsky, R. H. Middleton, and J. S. Freudenberg, "Feedback stabilization over signal-to-noise ratio constrained channels",, IEEE Transactions on Automatic Control, 52:1391-1403, 2007.

[2] N. Elia, "When Bode meets Shannon: Control-oriented feedback communication schemes," IEEE Transactions on Automatic Control 49(9):1477-1488, Sep 2004.

[3] S. A. A. Zaidi, T. Oechtering, and M. Skoglund, "Sufficient conditions for closed-loop control over multiple-access and broadcast channels," in *Proc. IEEE Conference on Decision and Control*, Atlanta, Georgia, Dec 2010.

[4] U.Kumar, J. N. Laneman and V. Gupta, "Cooperative Communications with Feedback via Stochastic Approximation," in *Proc. IEEE Information Theory Workshop*, Taormina, Sicily, Oct 2009.

[5] N. C. Martins and M. A. Dahleh, "Feedback Control in the Presence of Noisy Channels: Bode-Like Fundamental Limitations of Performance," IEEE Transactions on Automatic Control, 53(7):1604-1615, Aug. 2008.

[6] G. N. Nair and R. J. Evans, "Mean Square Stabilizability of Stochastic Linear Systems with Data Rate Constraints," in *Proc. IEEE Conference on Decision and Control*, 2002.

[7] A. Sahai and S. Mitter, "The Necessity and Sufficiency of Anytime Capacity for Stabilization of a Linear System over a Noisy Communication Link Part I: Scalar Systems," *IEEE Trans. on Information Theory*, 52(8):3369–3395, 2006.

[8] S. Tatikonda, and S. K. Mitter, "Control over noisy channels," IEEE Transactions on Automatic Control, 49(7):1196 - 1201, July 2004.

[9] J. P. M. Schalkwijk and T. Kailath, "A Coding Scheme for Additive Noise Channels with Feedback - I: No bandwidth constraint," *IEEE Trans. Inform. Theory*, vol. IT-12, pp. 172– 182, April 1966.