ANALYSIS OF CSIR FOR AN OFDM SYSTEM LIMITED BY A FREQUENCY-HOPPING INTERFERER

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Abstract

by

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This thesis explores an OFDM communication system in the presence of a unso-
plicated frequency-hopping interferer. Information theory is used to explore the fundamental limits of system performance with varying degrees of knowledge about the state of the interferer. In particular, the Shannon channel capacity of the system is derived for two different cases of channel state information at the receiver (CSIR), namely Perfect CSIR and Partial CSIR. An approximation of the Shannon channel capacity via Monte Carlo techniques is also made for the case of No CSIR for comparison. The gains in signal-to-noise ratio (SNR) obtained by Partial and Perfect CSIR for various interference powers are considered. The results show a trade-off between computational complexity and the gains in SNR.

Next, practical implementations of the system that approach the channel capacity are explored. The bit error rate (BER) of these systems is simulated as a function of SNR with various interference powers. Finally, an analysis is performed to compare the BER of the practical systems against the theoretic channel capacities found earlier. The results indicate that a practical system can obtain gains similar to those found from information theory.
To my wife, Caela,
and my son, David,
the joys in my life.
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CHAPTER 1

INTRODUCTION

1.1 Motivation and Problem Description

Orthogonal frequency division multiplexing (OFDM) appears to be the de-facto choice of modulation for wireless transmission in high-throughput systems. OFDM is used in current Wi-Fi standards and cellular standards such as LTE [1] [2]. As more wireless devices begin to crowd the spectrum, interference will undoubtedly begin to limit performance, whether intentional or not.

Previous work has considered OFDM systems with inter-carrier interference (ICI) [3], inter-block interference (IBI) [4], and co-channel interference (CCI) [5] (and the sources therein). Additionally, multiband OFDM systems in the presence of narrow-band interference [6] [7] and WiMAX interference [8] have been considered. Work has also been done on OFDM systems interfering with other OFDM systems [9].

In order to contribute to the investigation into the effects of interference on an OFDM system, this thesis analyzes the performance of an OFDM system in the presence of a frequency-hopping interferer.

We begin with a wireless OFDM system in a crowded or malicious environment in which interference is present. See Figure 1.1. Since thermal noise has a fairly flat power spectral density almost into the terahertz band for most temperatures [10 Sec. 4.5.2], we will consider the noise level to be constant, at least over short time intervals. With regard to fading, we will consider a stationary transmitter and receiver in an environment with no obstructions or reflectors. Consequently, we have slow
fading since our system is not in motion, and we have flat fading since no multipath propagation is present [11, 12]. Such conditions allow us to consider the effects of the interferer in the simplest context possible. In continuity with this mindset, we also assume that the interference is synchronous with our OFDM system. Synchronous interferers have been utilized in models previously [13], and asynchronous interference does not necessarily degrade performance more or less than synchronous interference [14]. Since the ambiguity is resolved by characteristics of the interferer that are not the focus of this thesis, we will utilize a synchronous interferer for tractability.

We will investigate the problem of an unsophisticated interferer that merely hops among the sub-carriers of the OFDM system. An example of such an unintentional interference may include a nearby system utilizing frequency-shift keying (FSK). Regarding malicious interference, we note that although the optimal interferer strategy from the view of information theory is to evenly spread energy in all degrees of free-
such a strategy is not often implemented. A simpler, albeit suboptimal,
interferer is a frequency hopper. Such an interference cannot simply be removed by
feedback to the transmitter. In fact, feedback to the transmitter is virtually worthless
in this case because the interference can move randomly with small dwell times.

Nonetheless, the receiver may be able to estimate characteristics of the interference with some training data. Of course, the signal processing involved to determine
certain characteristics will vary depending on the nature of what is to be determined.
A potential strategy that would offer some gain in throughput but not require as
much complexity would be to determine which sub-carrier in the OFDM system is
being effected by the interference.

This thesis explores these problems in detail and offer some insights. The results may be useful for stationary military communications in the desert near enemy
territory, as one example.

1.2 Contributions of the Thesis

The main contributions of this thesis are as follows:

- We derive the Shannon sum-channel capacity for our system of interest if perfect and partial channel state information are present at the receiver (CSIR). Additionally, we derive the optimal input distribution for the case of Partial CSIR.

- We explore ways to calculate the mutual information between random vectors with distributions that are Gaussian mixtures primarily for the case of No CSIR. In particular, we show that a Taylor series approximation is insufficient and verify Monte Carlo methods.

- We analyze the gains in signal-to-noise ratio (SNR) achieved by CSIR. In particular, we find that the ratio of the Partial CSIR gain to Perfect CSIR gain achieves a maximum for moderate interference powers. This suggests a strategy to differentiate which interference characteristics should be estimated for a given set of conditions.
• We show that an idealized implementation of our system of interest can approach the channel capacity within 0.8 to 2 dB in SNR. Consequently, the results can be used to inform the implementation of a practical system.

1.3 Outline

The remainder of thesis is organized as follows. Chapter 2 considers two high level abstractions of our system of interest in order to develop models for analysis. One model enables us to determine Shannon channel capacity through information theory. The other model enables us to analyze the bit-error rate performance of a system using practical applications of communication theory. Additionally, we provide the necessary background regarding information theory and communication theory that is used throughout the thesis.

After the models are introduced, Chapter 3 explores the fundamental limits of the system using information theory. We consider three cases of CSIR. First, we analyze Perfect CSIR and compute channel capacity. Then we calculate the channel capacity for Partial CSIR. Finally, we introduce methods to estimate the channel capacity for No CSIR. The results are then compared to demonstrate the gains in SNR available by determining the characteristics of the interference.

Next, Chapter 4 proceeds with analyzing the performance of an implementable communication system in the presence of an interferer. We describe in detail the implementation of the system and lay out the parameters for simulation. We then present the results of extensive simulations.

Finally, Chapter 5 discusses the relationship between the fundamental limits and the bit-error rates measured from our practical system. After calculating the precise relationships, we plot the results to show that a practical system can approach the fundamental limits. The relative size of the SNR gap between the performance of the practical system and the channel capacity indicates that the gains from CSIR found previously through information theory would be approximately applicable to
a practical system. Consequently, these results can inform future designs of real systems.
In this chapter, we present various levels of abstraction of our system of interest. The lower level abstraction is intended to be an idealization of a practical implementation of the system for which a bit-error rate can be measured. The higher level abstraction is meant to represent the system for analysis with information theory, with which we can compute fundamental limits of performance.

We begin with our high level abstraction of a continuous time OFDM system and evolve the model for our purposes.

2.1 Abstractions of the System

First, we consider a continuous-time abstraction of a channel with noise and interference as shown in Figure 2.1. Our aim is to develop an equivalent discrete-time OFDM model for the system.

![Figure 2.1. A basic continuous-time model of our system.](image-url)
2.1.1 OFDM System

We begin with a continuous-time model of an idealized OFDM system. For simplicity, we will assume perfect time synchronization, frequency synchronization, phase synchronization, and automatic gain control for the system. Additionally, we will assume ideal low-pass filters, analog-to-digital converters (ADC), and digital-to-analog converters (DAC). This simplified model is shown in Figure 2.2.

We will briefly explain the functions of the blocks and provide significantly more detail in Chapter 4. Note that solid lines in Figure 2.2 represent real signals while dashed lines denote complex-valued signals.

We begin with the data, represented in bits, that we want to send across the wireless channel. The data bits enter a look-up table (LUT) to select an appropriate

\[
\begin{aligned}
p &\approx 2 \cos (2\pi f_c t) \\
p &\approx 2 \sin (2\pi f_c t)
\end{aligned}
\]

(a) Transmitter

\[
\begin{aligned}
g(t) &\approx \sqrt{2} \cos (2\pi f_c t) \\
g(t) &\approx -\sqrt{2} \sin (2\pi f_c t)
\end{aligned}
\]

(b) Receiver

Figure 2.2. A continuous-time block diagram of the (a) transmitter and (b) receiver for a general OFDM system.
symbol from a given constellation, *e.g.* a Quadrature Amplitude Modulation (QAM) constellation. Consequently, the symbol can be represented by a complex number that we denote as $a \in \mathcal{A}$, where $\mathcal{A}$ is the constellation of possible symbols. The symbols proceed to a serial-to-parallel (S/P) conversion, where $M$ symbols are shifted in parallel for the $M$ sub-carriers of the OFDM system.

The $M$ parallel symbols then enter an $M$-point inverse fast Fourier transform (IFFT). Due to the nature of the IFFT, we are motivated to make $M$ a power of 2 to maximize performance. The IFFT acts as a pulse shaper, modulating the $M$ symbols onto frequency-shifted rectangular waveforms that are orthogonal [10]. The discrete-time outputs of the IFFT are effectively samples of the complex analog signal to be sent over the channel.

Next, the outputs of the IFFT are converted from parallel to serial (P/S), with the real and imaginary parts proceeding to separate digital-to-analog (DAC) converters. The real and imaginary parts of the signal are converted separately because ultimately we can only send a real signal through the channel. However, we will maintain the separation of real and complex signals by next modulating the real signal by a cosine function of a given frequency and the imaginary signal by an orthogonal sine function. These two modulated signals are added together to produce the real signal $x(t)$ to be sent over the channel.

The receiver structure in Figure 2.2b essentially inverts the functions of the transmitter. At the moment, we are not specifying the detector, though in general, we may say the detector approximates a maximum a posteriori (MAP) detector.

2.1.2 Channel

The frequency-hopping interferer can be thought of as an additional transmitter whose output signal $s(t)$ is added to $x(t)$. Additionally, noise will corrupt the signal
Figure 2.3. The channel adds the desired signal $x(t)$, interference $s(t)$, and noise $z(t)$ to produce the received signal $y(t) = x(t) + s(t) + z(t)$.

in the receiver. We will denote the additive noise process as $z(t)$. See Figure 2.3 for a visual relationship among the different signals.

2.1.3 Discrete-Time Baseband Equivalent System

Due to our assumptions regarding synchronization, ideal filters, DACs, and samplers, we can simplify Figure 2.2 further. We can ignore the up-conversion and down-conversion in continuous-time because our assumptions effectively state that the down-conversion process is an ideal inverse of the up-conversion process. As a result, we can simply consider the baseband equivalent model in discrete-time, shown in Figure 2.4. Note that we will have to consider the effects that this simplification has on our representation of the noise and interference. As we see, discrete complex values are being added in parallel to our signal of interest rather than a serial continuous-time signal. We will consider this in more depth later.

We can actually abstract the system in Figure 2.4 one level higher. We can ignore the particular symbol constellation and pulse-shaping and simply consider the discrete-time baseband channel represented by the summations in the center of Figure 2.4. Basically, we have $M$ orthogonal channels corresponding to the $M$
orthogonal sub-carriers. Each channel has one input and one output. The channel adds two quantities to the input, which creates uncertainty at the output.

Let us briefly introduce some new notation. Let the random variable \( X_i \) denote the input to channel \( i \), and the random variable \( Y_i \) denote the respective output. Let \( Z_i \) represent the value of discrete-time noise from the \( i \)th channel, and define the random variable \( S_i \) as the additive effect of the interference on the channel. With these definitions, we can abstract Figure 2.4 to be parallel channels, as shown in Figure 2.5. It is worth noting that if \( z(t) \) is modeled as stationary white Gaussian noise (SWGN), then the \( Z_i \) are independent, identically-distributed (iid) Gaussian random variables [16]. We also wish to note here that \( S_i \) will be modeled as correlated across frequency and memoryless over time to correspond to the behavior of the interferer.

Once again, we can abstract the model one level higher by modeling the channel as a conditional probability density given the input signal and interference signal as in Figure 2.6. This model will enable us to consider the benefits of having certain information about the interferer at the receiver, as we describe below.

In Figure 2.6, a codeword \( W \) is encoded to the vector signal \( \mathbf{X} = [X_1, X_2, \ldots, X_M]^T \) and transmitted over the channel, which can be modeled as a conditional probability density. We assume the interference signal vector, denoted \( \mathbf{S} \), has a probability
density $f_S(s)$. Information about the interference, denoted $U$ and $V$, is available to the transmitter and receiver, respectively. These values are also determined by a conditional probability density $f_{U,V|S}(u,v|s)$. The receiver measures $Y$ from the channel and gives an estimate of the transmitted codeword, denoted $\hat{W}$.

To be precise, we wish to make the distinction between knowledge of channel statistics and channel state information (CSI). CSI refers specifically to the realization of the state ($S = s$). Partial channel state information refers to some aspect of the state realization. For example, we describe knowing which OFDM sub-carrier the interference is affecting as partial CSI, even if the particular $S = s$ is not known. In our following analysis, we assume that the channel statistics (i.e. the densities $f_S(s)$, $f_{Y|X,S}(y|x,s)$, and $f_{U,V|S}(u,v|s)$) are always known at the transmitter and receiver. We will analyze the channel capacity for various scenarios of CSI at the receiver in the context of the model found in Figure 2.5, which we will now describe in more detail.
Figure 2.6. System block diagram capturing transmit and receive signal vectors $X$ and $Y$, respectively, the interference signal vector $S$, and the state information vector $U$ at the transmitter and $V$ at the receiver.

We model our OFDM system with $M$ sub-carrier as $M$ parallel channels. Each channel has continuous input $X_i$ and continuous output $Y_i$. Each channel also experiences additive white Gaussian noise, mathematically represented by the independent random variable $Z_i \sim \mathcal{N}(0, N_i)$[1] The frequency-hopping interference is assumed to have an additive effect modeled by the zero-mean Gaussian random variable $S_i$ with variance

$$\text{Var}(S_i) = \begin{cases} 0, & i \neq J \\ N_s, & i = J, \end{cases}$$

where $J$ is an integer between 1 and $M$. We simply model $J$ as a discrete random variable with distribution $p_J(j)$. Throughout this paper, we will assume $J$ is uni-

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1In this report, we use the notation $\mathcal{N}(\mu, \sigma^2)$ to denote the univariate Gaussian distribution with mean $\mu$ and variance $\sigma^2$. Similarly, with mean vector $\mu$ and covariance matrix $K$, $\mathcal{N}(\mu, K)$ denotes the multivariate Gaussian distribution.
formly distributed. Due to the unsophisticated nature of the interference, $J$ and the $S_i$ are independent of the $X_i$. However, an exception would occur in the case that CSI is available to the transmitter.

Following the development of Cover and Thomas [17, Ch. 9], we place a power constraint on the transmitter such that $E \left[ \sum_{i=1}^{M} X_i^2 \right] \leq P$. For notational convenience, our constraint can be written as $\sum_{i=1}^{M} P_i \leq P$. We will proceed with our analysis of the effects of CSI on the capacity of this channel in Chapter 3.

Now with our information theoretic models established, we continue by considering the metrics with which we will test our abstractions of the system.

2.2 Information Theoretic Metrics

For the models given by Figure 2.5 and Figure 2.6, we can utilize tools from information theory to derive fundamental limits in performance.

In general, we wish to measure the mutual information between the input $X$ and the output $Y$. As we will see in Section 2.4, mutual information is a function of the input distribution $f_X(x)$, over which we can optimize to obtain the sum channel capacity.

In this thesis, we will vary the amount of CSI available to the receiver and measure the performance of the system in terms of sum channel capacity. In cases for which the optimal input distribution is not clear, we offer an estimate of channel capacity.

2.3 Communication Theoretic Metrics

Next, let us consider our OFDM model in a more practical communication system, like that of Figure 2.2. However, we must make some adjustments in order to be able

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2Another potential scenario would be to model $J$ as a $K^{th}$-order Markov Process, resulting in channels with memory up to $K$ time instances. However, due to space considerations, this thesis will only consider a uniform $J$. 

13
to compare the performance of the system to the fundamental limits to be derived from information theory. In particular, we must utilize a capacity-approaching code. Consequently, we will add an encoder and decoder to the transmitter and receiver, respectively. A block diagram of the transmitter and receiver is in Figure 2.7. We will briefly explain the functions of the new blocks and provide more detail in Chapter 4.

We begin with the data, represented in bits, that we want to send across the channel. We first encode the data bits with a Low Density Parity Check (LDPC) code. The choice of an LDPC code is appropriate because the performance of the code enables the system to approach capacity, which will allow for a reasonable comparison between the systems in Figure 2.5 and Figure 2.7.

As before, the receiver structure in Figure 2.7b essentially inverts the functions of the transmitter. Notice that the input to the LDPC decoder are soft inputs from a

![Diagram of the transmitter and receiver](image)

Figure 2.7. A block diagram of the (a) transmitter and (b) receiver for our practical implementation of the system.
bit-wise log-likelihood ratio (LLR). More details on the implementation of the LLR will be given in Chapter 4.

Now we will consider the interference in more detail. Figure 2.8 shows our implementation of the interference transmitter in continuous-time. The production of $s(t)$ can be explained as a process similar to that in Figure 2.7a. We can skip the coding and LUT and begin with the inputs to the IFFT. Out of the group of $M$ symbols entering the IFFT, only one will be nonzero. The nonzero symbol will be complex Gaussian noise. This will allow the system to correspond to the behavior of Figure 2.5 with the interference having a dwell time equivalent to one OFDM symbol. The location of the nonzero symbol among the $M$ sub-carriers will be selected randomly from a uniform distribution. The interferer transmission proceeds as shown in Figure 2.8, which we described previously for the transmitter and will omit here.

As before, we will focus on the baseband-equivalent system in discrete time. Figure 2.9 shows the overall baseband-equivalent system with the noise and interference.

We will analyze the performance of this system by considering the bit error rate (BER) as a function of the so-called “signal-to-noise ratio per information bit,” denoted $E_b/N_0$. The relationship between BER as a function of $E_b/N_0$ and channel capacity as a function of SNR will be made explicit in Chapter 5 where the metrics of both abstractions are compared with each other.

Figure 2.8. A block diagram of the interferer transmitter.
Before we proceed further, we offer a brief review of the relevant topics and examples from information theory that will be useful to this thesis.

2.4 Background

Here we will go through basic definitions and establish the notation to be used throughout the thesis. Additionally, we provide examples of the Gaussian channel and parallel Gaussian channels to give a foundation on which to develop our results.

2.4.1 Definitions and Notation

We start with the definitions in information theory that will be useful to our analysis, taken from Cover and Thomas [17]. Entropy is a measure of the uncertainty of a random variable. For a continuous random variable $X$, we use differential entropy,
which is defined as

\[
h(X) = -\int_{\mathcal{X}} f(x) \log f(x) \, dx,
\]

(2.2)

where \( f(x) \) is the probability density function of \( X \) and \( \mathcal{X} \) is the support of \( f(x) \). Also note that all logarithms in this thesis have a base of 2, unless otherwise noted.

Given two continuous random variables \( X \) and \( Y \) with joint density \( f(x, y) \), the conditional entropy of \( X \) given \( Y \) is defined as

\[
h(Y|X) = -\int_Y \int_{\mathcal{X}} f(x, y) \log f(y|x) \, dx \, dy.
\]

(2.3)

These definitions of differential entropy are useful in calculating the mutual information between two continuous random variables \( X \) and \( Y \), which is given as

\[
I(X;Y) = \int_Y \int_{\mathcal{X}} f(x, y) \log \frac{f(x,y)}{f(x)f(y)} \, dx \, dy
\]

\[
= h(X) - h(X|Y)
\]

(2.4)

\[
= h(Y) - h(Y|X).
\]

(2.5)

(2.6)

Mutual information can be thought of as the amount of uncertainty removed from \( X \) due to knowledge of \( Y \).

In turn, the mutual information is crucial to determining the capacity of a channel. A channel’s Shannon capacity is the maximum rate of reliable transmission of information over the channel with an arbitrarily small probability of error [17, Ch. 7]. For the case of a memoryless channel, we compute capacity as the maximum mutual information that can result from varying the distribution of \( X \), or more formally,

\[
C = \max_{f(x)} I(X,Y).
\]

(2.7)
For this thesis, we wish to compute the capacity of a channel with interference as a means for determining bounds on the performance of a practical system with varying degrees of side information.

With the definitions given above, we present some useful results from Cover and Thomas [17, Ch. 2]. We begin with the chain rule for mutual information:

$$I(X_1, X_2, \ldots, X_n; Y) = \sum_{i=1}^{n} I(X_i; Y|X_1, X_2, \ldots, X_{i-1}).$$  \hspace{1cm} (2.8)

Equation (2.8) will be useful for evaluating the mutual information of a channel with multiple outputs. One such case occurs when the receiver knows the state of the channel, and the state can consequently be modeled as an additional output.

Another classic result that will be used in our work is the entropy of a multivariate normal distribution with mean vector $\mu$ and covariance matrix $K$. Given that $(X_1, X_2, \ldots, X_n) \sim \mathcal{N}_n(\mu, K)$, the entropy is given by

$$h(X_1, X_2, \ldots, X_n) = \frac{1}{2} \log \left( (2\pi e)^n |K| \right),$$  \hspace{1cm} (2.9)

where $|K|$ is the determinant of $K$.

Finally, Cover and Thomas present a theorem stating that a normal distribution maximizes the entropy of a random vector for a given variance. This result is useful in deriving the capacity of Gaussian channels. The derivation of equation (2.9) and the proof of the maximum entropy theorem can be found in [17, ch. 8].

We now provide two examples of channels along with their channel capacity.

2.4.2 The Gaussian Channel

First, we present the classic Gaussian channel, which is reproduced in Figure 2.10. The channel is a discrete-time, memoryless channel with continuous input
$X$, continuous output $Y$, and continuous noise $Z \sim \mathcal{N}(0,N)$. Hence, the channel probability density function $f_{Y|X}(y|x)$ is Gaussian with mean $x$ and variance $N$.

We will derive the capacity results here for the Gaussian channel (and for parallel Gaussian channels in Section 2.4.3) to demonstrate the rationale employed. With the addition of the interference later, we can utilize these simplifications only in some cases, which is to be demonstrated. We assume that the transmitter and receiver know the channel statistics (i.e. $Z \sim \mathcal{N}(0,N)$).

![Figure 2.10. The Gaussian channel.](image)

The capacity is given by (2.7) with the constraint that $f(x)$ must satisfy a power constraint $E[X^2] \leq P$. We will return to this issue in Section 2.4.3 for parallel Gaussian channels. First, we use (2.6) to obtain

\[
I(X;Y) = h(Y) - h(Y|X) \\
= h(Y) - h(X+Z|X) \\
= h(Y) - h(Z|X) \\
= h(Y) - h(Z), \tag{2.10}
\]
since differential entropy is invariant to a known shift and \( Z \) is independent of \( X \) \[17\] Ch. 8. Next, from (2.9) with \( n = 1 \), we find 
\[
h(Z) = \frac{1}{2} \log 2\pi eN.
\]
We also note that 
\[
\]
Since \( Z \sim \mathcal{N}(0, N) \) and 
\[
E[X^2] = P
\]
by our power constraint, 
\[
E[Y^2] = P + N.
\]
Recalling the theorem in Section 2.4.1 that the normal distribution maximizes the entropy of a random variable for a given variance, we know that 
\[
h(Y) \leq \frac{1}{2} \log 2\pi e(P + N)\] from (2.9). Hence,
\[
I(X; Y) = h(Y) - h(Z)
\]
\[
\leq \frac{1}{2} \log 2\pi e(P + N) - \frac{1}{2} \log 2\pi eN
\]
\[
= \frac{1}{2} \log \left( 1 + \frac{P}{N} \right),
\]
where equality occurs when \( X \sim \mathcal{N}(0, P) \). Consequently,
\[
C = \max_{f(x): E[X^2] \leq P} I(X; Y) = \frac{1}{2} \log \left( 1 + \frac{P}{N} \right),
\]
which is achieved if \( X \) is a zero-mean Gaussian random variable with variance \( P \).

2.4.3 Parallel Gaussian Channels

Here we consider the case in which we have \( M \) inputs and \( M \) outputs operating in parallel, each over an independent Gaussian channel. The collection of \( M \) channels is commonly called parallel Gaussian channels. Figure 2.11 illustrates parallel Gaussian channels.

Since the noise \( (Z_i) \) and inputs \( (X_i) \) are independent from channel to channel, the channels are said to be independent. As before, the channel statistics \( (Z_i \sim \mathcal{N}(0, N_i)) \) are known to the transmitter and receiver.

Let us simplify the notation by introducing vectors. We will say \( X = (X_1, X_2, \ldots, X_M) \), 
\( Z = (Z_1, Z_2, \ldots, Z_M) \), and \( Y = (Y_1, Y_2, \ldots, Y_M) \). The formula for capacity is analo-
Figure 2.11. Parallel Gaussian channels.

Consistent with that in (2.7):

\[ C = \max_{f(x): E \left[ \sum_{i} x_i^2 \right] \leq P} I(X; Y). \] (2.13)

As before, we expand the expression for mutual information following Cover and Thomas [17, Ch. 9]. Note that the power constraint is now written as \( E[X_i^2] = P_i \), and \( \sum P_i = P \).

\[ I(X; Y) = h(Y) - h(Y|X) \] (2.14)
\[ = h(Y) - h(X + Z|X) \] (2.15)
\[ \overset{a}{=} h(Y) - h(Z|X) \] (2.16)
\[ \overset{b}{=} h(Y) - h(Z) \] (2.17)
\[ \overset{c}{=} h(Y) - \sum h(Z_i) \] (2.18)
\[ \overset{d}{\leq} \sum h(Y_i) - h(Z_i) \] (2.19)
\[
\frac{e}{\epsilon} \leq \sum_i \frac{1}{2} \log \left( 1 + \frac{P_i}{N_i} \right).
\] (2.20)

Step (a) follows because differential entropy is invariant to a known shift [17, Th. 8.6.3].

Due to the independence of \(X_i\) and \(Z_i\), step (b) results. Steps (c) and (d) are true because \(h(W_1, W_2, \ldots, W_n) \leq \sum_{i=1}^n h(W_i)\), with equality if and only if the \(W_i\) are independent. Step (e) results because \(h(Z_i) = \frac{1}{2} \log 2\pi e N_i\) from equation (2.9) and \(h(Y_i) \leq \frac{1}{2} \log 2\pi e (P_i + N_i)\) by the theorem that the normal distribution maximizes entropy for a given variance. Hence, the capacity of parallel Gaussian channels is reduced to

\[
C = \max_{\sum_i P_i \leq P} \sum_i \frac{1}{2} \log \left( 1 + \frac{P_i}{N_i} \right),
\] (2.21)

which is achieved if the \(X_i\) are zero-mean Gaussian, each with variance \(P_i\). The \(P_i\) are chosen according to the water-filling method in which power is assigned beginning with the channel with the lowest noise and progressing to the channel with the second lowest noise, etc. Formally, \(P_i = (\nu - N_i)^+\), where \((w)^+ = w\) if \(w \geq 0\), or \((w)^+ = 0\) if \(w < 0\). The variable \(\nu\) is chosen such that \(\sum_i (\nu - N_i)^+ = P\).

It is worth noting that parallel channels can be used to model a system with multiple antennas or to model different time slots of communication within a system [18]. Therefore, any results derived with this model could be extended to these analogous situations.

Now let us proceed with the analysis of our abstractions, beginning with Figure 2.5 and our tools from information theory.
CHAPTER 3

FUNDAMENTAL LIMITS

In the scenarios described below, CSI at the receiver is denoted CSIR. For notational convenience, we will use subscripts to refer to a specific sub-carrier. To put Figure 2.5 in the context of Figure 2.6, we assume that the channel statistics \( f_{Y|X,S}(y|x,s) \) and \( f_S(s) \) (i.e. the distribution of the \( Z_i \) and \( S_i \)) are known to the transmitter and receiver. In other words, given \( X_i = x_i \) and \( S_i = s_i \), we know \( Y_i \sim \mathcal{N}(x_i + s_i, N_i) \). We can also say given \( X_i = x_i \) and \( J = j \), we know \( Y_i \sim \mathcal{N}(x_i, N_i + \text{Var}(S_i)) \), where the \( \text{Var}(S_i) \) is given in (2.1). Consequently, our scenarios of CSIR will be governed by what is known about \( S \).

3.1 Perfect CSIR Result

For the case of perfect CSIR, \( U = \emptyset \) and \( V = S \). In other words, the transmitter does not have any additional information, but the receiver knows the realization of the state \( S \). This CSIR can be treated as an additional output from the channel. Hence, to find the Shannon channel capacity, we are interested in maximizing the quantity \( I(X; Y, S) \) over the (continuous) input distribution \( f_X(x) \).  

First, we note \( I(X; Y, S) = I(X; S) + I(X; Y|S) \) by the chain rule of mutual information \[17 \text{ Th. 2.5.2}\]. Furthermore, because \( S \) is independent of \( X \), \( I(X; S) = 0 \) \[17 \text{ Eq. (2.90)}\]. Thus, we want to maximize the quantity \( I(X; Y|S) \). Let us simply express \( I(X; Y|S) \) with entropies.

\[1\] We will denote probability density functions with \( f(\cdot) \) and probability mass functions with \( p(\cdot) \).
\[ I(X; Y|S) = h(Y|S) - h(Y|X, S) \]
\[ \overset{a}{=} h(X + S + Z|S) - h(X + S + Z|X, S) \]
\[ \overset{b}{=} h(X + Z) - h(Z) \]
\[ \overset{c}{=} h(X + Z) - \sum_{i=1}^{M} h(Z_i) \]
\[ \overset{d}{\leq} \sum_{i=1}^{M} h(X_i + Z_i) - \sum_{i=1}^{M} h(Z_i) \]
\[ \overset{e}{=} \sum_{i=1}^{M} h(X_i + Z_i) - h(Z) \]
\[ \overset{f}{=} \sum_{i=1}^{M} h(X_i + Z_i) - \frac{1}{2} \log(2\pi e N_i) \]
\[ \overset{g}{\leq} \sum_{i=1}^{M} \frac{1}{2} \log(2\pi e (P_i + N_i)) - \frac{1}{2} \log(2\pi e N_i) \]
\[ \overset{h}{=} \sum_{i=1}^{M} \frac{1}{2} \log \left( 1 + \frac{P_i}{N_i} \right). \] (3.1)

Equality (a) results from \( Y = X + S + Z \). Since \( X \) and \( Z \) are independent of \( S \), we have equality (b). A general bound on entropy is \( h(R) \leq \sum_i h(R_i) \), with equality if and only if the \( R_i \) are independent [17, Th. 2.6.6]. This gives us (c) and (d). For (e), we simply changed the order of summation with the commutative property. Another useful property of entropy is that, for a zero-mean vector \( R \) with a given covariance matrix \( K \), \( h(R) \leq \frac{1}{2} \log((2\pi e)^n \det(K)) \), with equality if and only if \( R \sim \mathcal{N}(0, K) \) [17 Th. 8.6.5]. For the one-dimensional case, (f) and (g) result. Equality (h) is a simplification from the properties of logarithms.
Thus, \( I(X;Y|S) \leq \sum_{i=1}^{M} \frac{1}{2} \log \left( 1 + \frac{P_i}{N_i} \right) \). If \( X_i \sim \mathcal{N}(0, P_i) \), then this upper bound is realized. Hence, our optimization problem reduces to

\[
C = \max_{\sum_i P_i = P} \sum_{i=1}^{M} \frac{1}{2} \log \left( 1 + \frac{P_i}{N_i} \right),
\]

which can be solved following the arguments in [17, Sec. 9.4]. Ultimately, water-filling is used to determine the optimal power distribution.

We note that the result (3.2) is equivalent to the case in which no interference is present [17, Eq. 9.71]. We can intuitively interpret the result to mean that when the receiver knows the realization of the interference, the receiver simply removes the interference from the received signal.

### 3.2 Partial CSIR Result

A relevant case of partial CSIR includes \( U = \emptyset \) and \( V = J \). In other words, the transmitter does not have any additional information, but the receiver knows which sub-carrier experiences the interference. Since the realization of \( S \) is not known, \( V = J \) is said to be only partial state information. As before, we treat the additional information at the receiver as another output of the channel:

\[
I(X;Y,J) = I(X;J) + I(X;Y|J)
\]

\[
\overset{a}{=} I(X;Y|J)
\]

\[
\overset{b}{=} \sum_{j=1}^{M} p_J(j) I(X;Y|J = j).
\]

Here, equality \( (a) \) follows because \( X \) is independent of \( J \). Equality \( (b) \) is the law of total probability. We now consider \( I(X;Y|J = j) \) in terms of entropies. For convenience we will define \( \tilde{Z} = Z + S \), which we note has conditionally independent
components $\tilde{Z}_i \sim \mathcal{N}(0, \tilde{N}_i)$ given $J = j$, where $\tilde{N}_j = N_j + N_s$, and $\tilde{N}_i = N_i, \forall i \neq j$.

Hence,

\[
I(X; Y | J = j) = h(Y | J = j) - h(Y | X, J = j) \\
= h(X + \tilde{Z} | J = j) - h(X + \tilde{Z} | X, J = j) \\
= h(X + \tilde{Z} | J = j) - h(\tilde{Z} | J = j) \\
= h(X + \tilde{Z} | J = j) - \sum_{i=1}^{M} h(\tilde{Z}_i | J = j) \\
\leq \sum_{i=1}^{M} h(X_i + \tilde{Z}_i | J = j) - h(\tilde{Z}_i | J = j) \\
= \sum_{i=1}^{M} h(X_i + \tilde{Z}_i | J = j) - \frac{1}{2} \log \left( 2\pi e \tilde{N}_i \right) \\
\leq \sum_{i=1}^{M} \frac{1}{2} \log \left( 2\pi e (P_i + \tilde{N}_i) \right) - \frac{1}{2} \log \left( 2\pi e \tilde{N}_i \right) \\
= \sum_{i=1}^{M} \frac{1}{2} \log \left( 1 + \frac{P_i}{\tilde{N}_i} \right) \\
= \frac{1}{2} \log \left( 1 + \frac{P_j}{N_j + N_s} \right) + \sum_{i=1, i \neq j}^{M} \frac{1}{2} \log \left( 1 + \frac{P_i}{\tilde{N}_i} \right). \quad (3.4)
\]

The process of equalities and inequalities follows the development given previously for (3.1). Note that the bound can be reached if $X_i \sim \mathcal{N}(0, P_i)$.

Finally, we substitute the results from (3.4) into (3.3) to obtain the capacity.

\[
C = \max_{\sum_i P_i = P} I(X; Y, J) \\
= \max_{\sum_i P_i = P} \sum_{j=1}^{M} p_j(j) \left[ \frac{1}{2} \log \left( 1 + \frac{P_j}{N_j + N_s} \right) + \sum_{i=1, i \neq j}^{M} \frac{1}{2} \log \left( 1 + \frac{P_i}{\tilde{N}_i} \right) \right]
\]
\[
E_J \left( \sum_{i \neq J} P_i, P_J \right) = \max_{\sum_i P_i = P} \left[ \frac{1}{2} \log \left( 1 + \frac{P_J}{N_J + N_s} \right) + \sum_{i=1}^{M} \frac{1}{2} \log \left( 1 + \frac{P_i}{N_i} \right) \right].
\] (3.5)

If \( J \) is uniformly distributed, then (3.5) simplifies to

\[
C = \max_{\sum_i P_i = P} \sum_{j=1}^{M} \left[ \frac{1}{2} \log \left( 1 + \frac{P_j}{N_j + N_s} \right) + \sum_{i=1}^{M} \frac{1}{2} \log \left( 1 + \frac{P_i}{N_i} \right) \right].
\] (3.6)

The optimal power distribution for (3.6) is found once again via the water-filling technique. An extensive proof for this result can be found in Appendix A.

3.3 No CSIR Result

With no CSI available to the transmitter or receiver (\( U = V = \emptyset \)), the general formula for capacity holds in which there is no additional output and the optimization occurs over \( f_X(x) \):

\[
C = \max_{\sum_i P_i \leq P} I(X; Y).
\] (3.7)

Here \( I(X; Y) = h(Y) - h(Y|X) = h(X + S + Z) - h(S + Z) \). Further analysis is difficult because \( f_S(s) \) is a Gaussian mixture of the form \( \frac{1}{M} \sum_{j=1}^{M} \mathcal{N}(0, K(j)) \), where \( K(j) \) is an all-zero matrix except for the entry \((j, j)\) which equals \( N_s \).

Several techniques exist for determining the entropy of Gaussian mixtures. For instance, Kim et al. approximate a large Gaussian mixture in a local neighborhood by considering only the nearest Gaussian components that will have a significant impact on their calculations and ignoring the rest [19]. Several approximations exist for MIMO channels, as well [20]–[22]. We will use the technique from Huber et al. [23], in which the Taylor Series expansion of the natural logarithm function is used.
to simplify the calculation of the entropy of a Gaussian mixture random vector. Essentially, by expanding the logarithm component-wise around the mean of a given Gaussian component, the integrations required for entropy result in finding the central moments of a Gaussian.

3.3.1 Second Order Taylor Series Approximation

Mathematically, Huber et al. defines a Gaussian mixture of $L$ Gaussian components to be of the form

$$f(x) = \sum_{i=1}^{L} \omega_i \cdot \mathcal{N}(x; \mu_i, C_i),$$

(3.8)

where $\omega_i > 0$ and $\sum_{i=1}^{L} \omega_i = 1$. The notation $\mathcal{N}(x; \mu_i, C_i)$ denotes that the multivariate distribution is a function of $x$.

The entropy (in nats) of the random vector $x$ is given by

$$h(X) = -\int_{\mathbb{R}^N} f(x) \log f(x) \, dx.$$  

(3.9)

Huber et al. [23, Eq. (4)] approximates (3.9) by using an $R$th order Taylor series expansion of the natural logarithm and summing over all components:

$$h(X) \approx -\sum_{i=1}^{L} \int_{\mathbb{R}^N} \omega_i \cdot \mathcal{N}(x; \mu_i, C_i) \cdot \left( \sum_{k=0}^{R} \frac{1}{k!} \left( (x - \mu_i) \odot \tilde{\nabla} \right)^k \log f(\tilde{x}) \right) \Bigg|_{\tilde{x}=\mu_i} \, dx,$$

(3.10)

where $\tilde{\nabla}$ is the gradient with respect to $\tilde{x}$ and $\odot$ is the matrix contradiction operator, which consists of an element-wise matrix multiplication followed by a summation of all elements. Mathematically, $C \odot D = \sum_i \sum_j C_{ij} D_{ij}$.
The approximation (3.10) can be solved analytically since evaluating the integral involves determining the first $R$ central moments of the Gaussian components. Fortunately, the odd central moments of a Gaussian are zero. For tractability, we will use the second order Taylor series expansion to approximate the entropy of our Gaussian mixture random vectors. The approximation is given by

$$h_2(X) \approx h_0(X) - \sum_{i=1}^{L} \frac{\omega_i}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \eta(i, j, k) C_{ijk},$$  \hspace{1cm} (3.11)

where $h_0(X)$ is the zeroth order approximation of differential entropy derived in Appendix B.1, $\eta(i, j, k)$ is given in Appendix B.2, and $C_{ijk}$ refers to the element in the $j$th row and $k$th column of the covariance matrix of the $i$th Gaussian component.

With the approximation in (3.11), we return to the problem of solving $h(Y) - h(Y|X)$. We make the assumption that the optimal solution will involve a zero-mean Gaussian input vector $X$. This assumption is reasonable given that a Gaussian distribution maximizes entropy for a given second order constraint, which is present in our case as a power constraint on the input. As we have seen in the cases for Perfect and Partial CSIR, the upper bound for information rate was achieved through a Gaussian $X$ for this reason.

With the assumption of a Gaussian $X$ and conditioned on knowledge of $J = j$, we can see $Y_i|J = X_i + Z_i + S_i|J = X_i + \tilde{Z}_i|J$. Therefore, $Y_i|J \sim \mathcal{N}(0, P_i + \tilde{N}_i)$. In

\footnote{Note that the result of (3.11) is in nats. To convert to bits, we need to multiply the result by $\log_2 e$.}
vector notation,

\[
f_{y|j}(y|j) = \mathcal{N} \left( y; 0, \begin{bmatrix} N_1 + P_1 & \cdots & \cdots & \cdots \\ \vdots & \ddots & \cdots & \cdots \\ \cdots & \cdots & N_j + P_j + N_s & \cdots \\ \cdots & \cdots & \cdots & N_M + P_M \end{bmatrix} \right). \tag{3.12}
\]

Thus, by the law of total probability, we see that \( f_Y(y) \) is a Gaussian mixture:

\[
f_Y(y) = \sum_{j=1}^{M} p_J(j) f_{y|j}(y|j). \tag{3.13}
\]

A similar result occurs for \( Y|X \). To begin, we note that \( h(Y|X) = h(X+Z+S|X) = h(Z+S) = h(\tilde{Z}) \). If we condition \( \tilde{Z} \) on \( J = j \), we have

\[
f_{\tilde{z}|j}(\tilde{z}|j) = \mathcal{N} \left( \tilde{z}; 0, \begin{bmatrix} N_1 & \cdots & \cdots & \cdots \\ \vdots & \ddots & \cdots & \cdots \\ \cdots & \cdots & N_j + N_s & \cdots \\ \cdots & \cdots & \cdots & N_M \end{bmatrix} \right). \tag{3.14}
\]

Once again, by the law of total probability, we find \( f_{\tilde{z}}(\tilde{z}) \) is a Gaussian mixture:

\[
f_{\tilde{z}}(\tilde{z}) = \sum_{j=1}^{M} p_J(j) f_{\tilde{z}|j}(\tilde{z}|j). \tag{3.15}
\]

Consequently, both \( h(Y) \) and \( h(Y|X) = h(\tilde{Z}) \) can be evaluated with (3.11), and the mutual information \( I(X;Y) \) can be approximated.
Now that we have presented the approximation, we need to determine the accuracy of the approximation. As a result, we need to calculate the true value of the differential entropy. Since an analytical solution appears elusive, we will use numeric integration. In order to use numeric integration in a timely fashion, we will have to reduce the number of dimensions in our problem. To maintain a Gaussian mixture, though, we need to keep $M \geq 2$. Consequently, we begin with $M = 2$ and extend to $M = 3$ and 4.

3.3.1.1 Comparison Between Approximation and True Values

Here we will compute the mutual information between the system inputs and outputs both via Huber’s second-order Taylor Series approximation and via numeric integration. In this example, we have $M = 2$ and vary the total input power $P$. We consider the cases of $N_s = 1, 5, 10, \text{ and } 16$ watts. We also plot the channel capacities for Partial and Perfect CSIR for completeness. The results are in Figure 3.1.

It is obvious that the Taylor Series approximation has varied results. For $N_s = 1$ watt, the approximation is decently close to the true value. However, as interference power ($N_s$) increases, the Taylor Series approximation becomes worse and departs further from the true value. In fact, for $N_s = 10$ and 16, the Taylor Series approximation gives a useless result that the mutual information is negative when by definition $I(X;Y) \geq 0$.

For the sake of space, we will omit the figures for $M = 3$ and $M = 4$, though the results are qualitatively the same. The Taylor Series approximation departs further from the true value as interference power increases. As a summary, we provide Figure 3.2 which is a graph of the difference between the Taylor Series approximation

\footnote{Numeric integration was accomplished through MATLAB’s ‘\texttt{integral}’ function. See http://www.mathworks.com/help/matlab/ref/integral.html for more details.}
Figure 3.1. A comparison of the second-order Taylor Series approximation of mutual information with the true value for \( M = 2 \) parallel channels. The sum mutual information is given as a function of SNR for systems with Perfect, Partial, and No CSIR. The interference power \( N_s \) is varied.

and the true mutual information as a function of \( N_s \) for \( M = 2, 3, \) and 4 sub-carriers. The data points are taken at \( P/M = 0 \) dB.

Note that the difference between the approximation and true value generally increase with \( N_s \) and \( M \). Consequently, we will consider a new approach for approximating the true mutual information that is computationally feasible for a 16-dimensional integral. In particular, we will consider the use of Monte Carlo techniques.
3.3.2 Monte Carlo Approximation

First, we will offer a justification for the veracity of the results obtained from Monte Carlo techniques. Then we will proceed to describe the algorithm we used to obtain our results.

3.3.2.1 Justification

The law of large numbers (LLN) states in general that averaging over the results of a large number of trials of an experiment should approach the true expected value [24 Sec. 14.3]. The average will tend to the expected value as the number of trials increases.
Let us define the sample mean of \( n \) trials as

\[
\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i,
\]

where \( X_i \) is the result of a single trial.

Two versions of the law of large numbers exist, differing in their strength of proof for convergence. The \textit{weak law of large numbers} states that for independent, identically distributed (iid) random variables \( X_i \) with a finite mean, denoted \( E[X_i] \), the sample mean will converge in probability to \( E[X_i] \) for large \( n \). In other words,

\[
\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \rightarrow E[X_i]
\]

in probability \cite[Sec. 14.3]{24}.

The \textit{strong law of large numbers} states that for iid \( X_i \) with a finite mean, the sample mean \( \frac{1}{n} \sum_{i=1}^{n} X_i \rightarrow E[X_i] \) almost surely for large \( n \), which is a stronger statement of convergence. Proof for both statements can be found in \cite[pp. 116-117]{24} and \cite[pp. 326-331]{25}, respectively.

In information theory, we often want to compute the mutual information between two random variables, \( X \) and \( Y \). Recall that mutual information is mathematically defined as

\[
I(X;Y) = \int_{\mathcal{Y}} \int_{\mathcal{X}} f_{XY}(x,y) \log_2 \left( \frac{f_{XY}(x,y)}{f_X(x)f_Y(y)} \right) \, dx \, dy
\]

\[
= E_{f_{XY}(x,y)} \left[ \log_2 \left( \frac{f_{XY}(x,y)}{f_X(x)f_Y(y)} \right) \right],
\]

where \( \mathcal{X} \) and \( \mathcal{Y} \) are the set of support for \( f_{XY}(x,y) \), and the expectation is taken with respect to \( f_{XY}(x,y) \). We can also compute the mutual information between
random vectors, $\mathbf{X}$ and $\mathbf{Y}$, by using a similar formula with random vectors replacing the random variables.

For complicated distributions, analytical solutions of (3.18) are elusive, so numerical integration is appealing. However, the curse of dimensionality makes numeric integration incredibly time-consuming and impractical for vectors of more than five dimensions. Consequently, for large dimensional problems, Monte Carlo methods are preferred. The Monte Carlo method utilizes the law of large numbers by generating a large number of sample points and averaging the results to approximate the expected value.

In the case of estimating (3.18), we generate $n$ iid samples $x_i$ and $y_i$ from $f_{XY}(x, y) = f_X(x)f_{Y|X}(y|x)$ and calculate

$$\frac{1}{n} \sum_{i=1}^{n} \log_2 \left( \frac{f_{XY}(x_i, y_i)}{f_X(x_i)f_Y(y_i)} \right),$$  

(3.19)

which should converge to $I(X; Y)$ for large $n$ according to the law of large numbers.

Hence, using Monte Carlo simulation to approximate mutual information is justified by the law of large numbers.

### 3.3.2.2 Procedure

We wish to use the method described above to approximate the mutual information between the input and output of our system of interest with no channel state information at the receiver.

As a consequence of the interferer, the output $Y_i = X_i + Z_i + S_i$ has a distribution that is a Gaussian mixture. Please refer to equations (3.12) and (3.13) for proof.
Since \( f_{XY|J}(x, y|j) = f_X(x) f_{Y|X,J}(y|x, j) \) and \( x \) can be generated independently, it is worth noting that \( f_{Y|X,J}(y|x, j) \) is Gaussian:

\[
f_{Y|X,J}(y|x, j) = N \left( y; x, \begin{bmatrix} N_1 \\ \vdots \\ N_j + N_s \\ \vdots \\ N_M \end{bmatrix} \right).
\] (3.20)

By the law of total probability, we find \( f_{Y|X}(y|x) \) is a Gaussian mixture:

\[
f_{Y|X}(y|x) = \sum_{j=1}^{M} p_J(j) f_{Y|X,J}(y|x, j).
\] (3.21)

Since conditioning on \( J \) greatly simplifies the distribution of \( Y \) and \( Y|X \), and generating normal random vectors is simple in comparison to generating random vectors from a Gaussian mixture, we will draw our sample vectors for the Monte Carlo method from \( f_{XY|J}(x, y|j) = f_X(x) f_{Y|X,J}(y|x, j) \) instead of \( f_{XY}(x, y) \).

As a consequence, the expectation must be taken with respect to \( f_{XY|J}(x, y|j) \). In other words, if we generate iid sample vectors \( x_k \) and \( y_k \) from \( f_X(x) f_{Y|X,J}(y|x, j) \), we will have

\[
\frac{1}{n} \sum_{k=1}^{n} \log_2 \left( \frac{f_{Y|X,J}(y_k|x_k)}{f_Y(y_k)} \right) \rightarrow E_{f_{XY|J}(x,y|j)} \left[ \log_2 \left( \frac{f_{Y|X,J}(y|x)}{f_Y(y)} \right) \right] \bigg| J = j. \] (3.22)
Thus, we will again need to use the law of total probability to obtain the expected value (mutual information):

\[
E \left[ \log_2 \left( \frac{f_{Y|X}(y|x)}{f_Y(y)} \right) \right] = \sum_{j=1}^{M} E \left[ \log_2 \left( \frac{f_{Y|X}(y|x)}{f_Y(y)} \right) \bigg| J = j \right] p_J(j). \tag{3.23}
\]

We will outline the necessary steps to calculate the approximate mutual information via the Monte Carlo method below. Note that mutual information is simply the mean of \( \log_2 \left( \frac{f_{Y|X}(y|x)}{f_Y(y)} \right) \), so the Monte Carlo method gives a sample mean or a sample mutual information.

To compute an approximation of the mutual information as a sample mean, we first generate \( n/M \) sample vectors \( x_k \) and \( y_k \) from the distribution \( f_{X,Y|J}(x,y|j) = f_X(x)f_{Y|X,J}(y|x,j) \) for a given \( J = j \). Since the \( M \) possible values of \( j \) are uniformly distributed, we generate \( n/M \) sample vectors for each value of \( J = j \), which results in a total of \( n \) sample vectors. Since \( f_X(x) \) and \( f_{Y|X,J}(y|x,j) \) are both multivariate Gaussian, this step is efficient and simple.

Second, we must compute the approximation to the expected value of the logarithm given \( J \), given by (3.22). To do so, we first plug our sample vectors \( x_k \) and \( y_k \) into \( f_{Y|X,J}(y|x,j) \) and \( f_{Y|J}(y|j) \). With the system set-up, the output components are uncorrelated, which allows us to write

\[
f_{Y|X,J}(y|x,j) = \prod_{l=1}^{M} \frac{1}{\sqrt{2\pi C_{Y|X,J}(l,l)}} \exp \left( -\frac{1}{2} \frac{(y_l - x_l)^2}{C_{Y|X,J}(l,l)} \right) \tag{3.24}
\]

\[
f_{Y|J}(y|j) = \prod_{l=1}^{M} \frac{1}{\sqrt{2\pi C_{Y|J}(l,l)}} \exp \left( -\frac{1}{2} \frac{y_l^2}{C_{Y|J}(l,l)} \right), \tag{3.25}
\]

where \( C(j,k) \) refers to the element in the \( j \)th row and \( k \)th column of the covariance matrix \( C \). The advantage of calculating the values of \( f_{Y|X,J}(y_k|x_k,j) \) and \( f_{Y|J}(y_k|j) \)
in this way lies in the ability to leverage MATLAB’s speed with vector operations across the large number of samples.\footnote{For instance, performing 16 iterations of a vectorized formula with 1,000,000 elements in MATLAB is faster than performing 1,000,000 iterations of a vectorized formula with 16 elements.}

Next, we utilize the law of total probability to calculate

\[
f_{Y|X}(y_k|x_k) = \sum_{j=1}^{M} p_J(j) f_{Y|X,J}(y_k|x_k, j) \quad (3.26)
\]

\[
f_Y(y_k) = \sum_{j=1}^{M} p_J(j) f_{Y|J}(y_k|j). \quad (3.27)
\]

Then we compute an approximation of \(E[\log_2(f_{Y|X}(y|x)/f_Y(y))|J = j]\) with

\[
\frac{1}{n} \sum_{k=1}^{n} \log_2 \left( \frac{f_{Y|X}(y_k|x_k)}{f_Y(y_k)} \right).
\quad (3.28)
\]

Finally, we use the result of the approximation above to compute

\[
E \left[ \log_2 \left( \frac{f_{Y|X}(y|x)}{f_Y(y)} \right) \right] = \sum_{j=1}^{M} E \left[ \log_2 \left( \frac{f_{Y|X}(y|x)}{f_Y(y)} \right) \middle| J = j \right] p_J(j). \quad (3.29)
\]

Since this mean corresponds to the mutual information \(I(X;Y)\), our result is an approximate mutual information.

Computing an approximation for the variance of \(\log_2(f_{Y|X}(y|x)/f_Y(y))\) is very similar to the process of computing an approximation for the mean. First, we generate sample vectors \(x_k\) from \(f_X(x)\) and \(y_k\) from \(f_{Y|X,J}(y|x, j)\) so that the samples are iid from \(f_{XY|J}(x, y|j)\). Once again, we generate \(n/M\) sample vectors for each of the \(M\) values of \(j\). This results in \(n\) total sample vectors.
Next, we want to make the approximation

\[
\frac{1}{n} \sum_{k=1}^{n} \left( \log_2 \left( \frac{f_{Y|X}(y_k|x_k)}{f_Y(y_k)} \right) \right)^2 \rightarrow E \left[ \left( \log_2 \left( \frac{f_{Y|X}(y|x)}{f_Y(y)} \right) \right)^2 \bigg| J = j \right] \quad (3.30)
\]

via the law of large numbers. To do so, we use the sample vectors to evaluate (3.24) and (3.25), which we can then use to evaluate (3.26) and (3.27), respectively. With values for \( f_{Y|X}(y_k|x_k) \) and \( f_Y(y_k) \), we can compute the approximation given by (3.30).

Then we use the law of total probability to compute

\[
E \left[ \left( \log_2 \left( \frac{f_{Y|X}(y|x)}{f_Y(y)} \right) \right)^2 \bigg| J = j \right] = \sum_{j=1}^{M} p_J(j) E \left[ \left( \log_2 \left( \frac{f_{Y|X}(y|x)}{f_Y(y)} \right) \right)^2 \bigg| J = j \right] \quad (3.31)
\]

from the approximation. The final step is to calculate the approximate variance from the first and second moments:

\[
\text{VAR} \left( \log_2 \left( \frac{f_{Y|X}(y|x)}{f_Y(y)} \right) \right) = E \left[ \left( \log_2 \left( \frac{f_{Y|X}(y|x)}{f_Y(y)} \right) \right)^2 \right] - E \left[ \log_2 \left( \frac{f_{Y|X}(y|x)}{f_Y(y)} \right) \right]^2 .
\]

Note that the first moment is calculated using the procedure given previously.

### 3.4 Simulation Results

As an example, let us consider an OFDM system with \( M = 16 \) sub-carriers. Let \( J \) be uniformly distributed on \( \{1, 2, \ldots, 16\} \) watts. We set the noise power equivalently across all sub-channels as \( N_i = 1, \forall i \). Finally, we vary the interference power with \( N_s = 1, 5, 10, \) and \( 16 \). Figure 3.3 shows the capacity (or approximation of mutual information) as a function of the total transmit power \( P \) for each case.
Figure 3.3. Channel capacity (or an approximation) as a function of SNR for systems with Perfect, Partial, and No CSIR. The interference power $N_s$ is varied. Note that the curve for No CSIR is obtained via Monte Carlo simulation described in Section 3.3.2.

Let us make some remarks about Figure 3.3. The $y$-axis measures the sum-capacity of the $M$ parallel channels that are modeling the $M$ sub-carriers of an OFDM system. The units are bits per channel use. The $x$-axis measures the total transmit power $P$ divided by the total number of sub-carriers $M$. For this simulation, the power of the noise in each channel is equal to 1 W. Since the optimal power
distribution at the transmitter is water-filling against the noise power. The transmit power for each sub-carrier in this example is $P/M$. Since the noise power is 1 W, we can write $P/M$ in terms of decibels and think of $P/M$ as SNR for an individual sub-carrier. In other words, $\text{SNR} = P_i/N_i = P_i$, and since the power distribution is uniform in this case $P_i = P/M$.

As a result, we can measure the gains that CSIR garners as a function of the interference power. The gains achieved by CSIR are listed in Table 3.1 measured at 14 bits/channel use. The gains were computed via linear interpolation from the data in Figure 3.3. In particular, the Perfect CSIR gain was measured from the No CSIR curve to the Perfect CSIR curve, and the Partial CSIR gain was measured similarly.

As expected, the system with Perfect CSIR has a higher throughput than the system with Partial CSIR. Similarly, both have a higher capacity than that of the system with No CSIR. Hence, the results do not conflict with our general expectations. We will analyze the results more thoroughly next.

3.5 Analysis

Note that the gain for both perfect and partial CSIR increase with increasing interference power ($N_s$). This makes sense because a larger interference power will deteriorate system performance more than a lower interference power. Consequently, knowledge about the interference is more useful when the jammer is powerful and more detrimental to the system performance. Thus, the gains from CSIR increase with increasing interference power.

Next, we want to note that perfect CSIR provides the maximum possible gain from CSIR, whereas partial CSIR provides only a fraction of the total possible gain.

\[ \text{This was proven for the cases of Perfect CSIR and Partial CSIR. It was assumed true for the case of No CSIR since the transmitter has the same channel state information as the other cases.} \]
TABLE 3.1

THE SNR GAINS FROM CSIR FOR \( M = 16 \) SUB-CARRIERS AT 14 BITS/CHANNEL USE.

<table>
<thead>
<tr>
<th>( N_s ) (W)</th>
<th>Partial CSIR gain (dB)</th>
<th>Perfect CSIR gain (dB)</th>
<th>Percent of Total Gain Achieved by Partial CSIR (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.090</td>
<td>0.262</td>
<td>34.26</td>
</tr>
<tr>
<td>5</td>
<td>0.460</td>
<td>0.813</td>
<td>56.58</td>
</tr>
<tr>
<td>10</td>
<td>0.605</td>
<td>1.015</td>
<td>59.60</td>
</tr>
<tr>
<td>12</td>
<td>0.624</td>
<td>1.045</td>
<td>59.68</td>
</tr>
<tr>
<td>14</td>
<td>0.633</td>
<td>1.064</td>
<td>59.55</td>
</tr>
<tr>
<td>16</td>
<td>0.635</td>
<td>1.072</td>
<td>59.21</td>
</tr>
<tr>
<td>20</td>
<td>0.630</td>
<td>1.077</td>
<td>58.47</td>
</tr>
</tbody>
</table>

Nonetheless, determining the realization of \( S = s \) is undoubtedly more complex than determining \( J = j \), so a trade-off exists between the system complexity and gain. With this in mind, it is interesting to note that the gain from Partial CSIR as a percentage of the possible gain varies with interference power. In other words, the gain from Partial CSIR as a fraction of the gain from Perfect CSIR reaches a maximum value for a certain value of \( N_s \). Our simulations indicate that this value is around \( N_s = 12 \) watts, for which Partial CSIR achieves about 60% of the potential gain, but more work needs to be done to make sure that these values are accurate and not simply an effect from linear interpolation. Nonetheless, the results indicate that Partial CSIR achieves a larger percentage of the potential gain for moderate values of \( N_s \), whereas Partial CSIR attains a smaller percent of the potential gain when the interference power is low (\( N_s = 1 \) W) or extremely high (\( N_s = 100 \) W).
An interesting question to pose might be the following: Why is the capacity gap between the Perfect (and Partial) CSIR and No CSIR in Figure 3.3 smallest at $P/M = 0$ dB? The interferer power relative to the transmit power is strongest at this point. It would seem that the gap should be largest at $P/M = 0$ dB.

We will attempt here to provide both a qualitative and an analytical explanation. We will use Figure 3.4 to offer some intuition as to why the gap in capacity is smallest at $P/M = 0$ dB. The figures represent different relative transmitter and interference powers for the sub-carrier on which the interference is present. Let us fix the interference power $N_s$ to be constant across all the cases in Figure 3.4.

First, let us consider Figure 3.4a. The interference power is much greater than the transmission power. Consequently, the ability to “strip off” the interference with Perfect CSIR will gain us the entire transmission power without interference.

A similar situation occurs in Figure 3.4b but the transmission power is closer to the value of the interference power. Again, stripping off the interference here will gain us the entire transmission power without interference. Since the transmission power in Figure 3.4b is greater than the transmission power in Figure 3.4a the gain is correspondingly larger.

Thus, the gap between the Perfect CSIR and No CSIR curves became larger for a greater $P/M$ against a constant $N_s > P/M$. In simple terms, the reason is that a larger $P/M$ offers more transmission power with which to interfere. Therefore, removing the interference for a larger $P/M$ offers more gain. Note that this is only the case for $N_s > P/M$, as we will explore next.

Figure 3.4c shows the case in which the transmission power is much larger than the interference power. Obviously, stripping off the interference offers a gain in throughput as it always will. Instead of utilizing power to overcome the interference, the transmitter can apply that same power to transmission with Perfect CSIR.
Figure 3.4. A comparison of the sub-carrier transmission power \( (P/M) \) and the interference power \( (N_s) \). When \( N_s > P/M \), the reduction in throughput caused by the interferer increases with \( P/M \) as demonstrated by (a) and (b). When \( P/M \gg N_s \), the SNR gain from stripping off the interference remains approximately the same, as demonstrated in (c) and (d).

Looking now at Figure 3.4d, we see the transmission power increase more. Again, removing the interference offers a gain. The transmission power used to overcome the interference can be used for transmitting throughput. In comparing Figure 3.4c with Figure 3.4d, we can see that the gain from Perfect CSIR would be approximately the same.

Thus, for high values of \( P/M \) relative to \( N_s \), the gap between the Perfect CSIR and No CSIR curves will be approximately equivalent. Consequently, we modify our
simple qualitative rule. When the interference power is greater than the transmission power \( (N_s > P/M) \), the gap increases with increasing \( P/M \) because a larger \( P/M \) offers more power with which to interfere. When the interference power is less than the transmission power \( (N_s < P/M) \), the gap approaches a constant value with increasing \( P/M \) because the maximum power with which to interfere has already been reached.

This qualitative argument is not meant to be a rigorous proof. Nonetheless, the explanation gives some intuition for the relative size of the gap between the Perfect CSIR and No CSIR curves.

Next, we offer a more analytical explanation for the size of the gap between the Perfect CSIR and Partial CSIR curves. For the case in which the interferer’s frequency hops are uniformly distributed across the sub-carriers \( (i.e., J \sim \text{uniform}[1, M]) \), the sum-channel capacity for the cases of Perfect CSIR and Partial CSIR, respectively, are

\[
C_{\text{Perfect}} = \max_{\sum_i P_i = P} \frac{1}{M} \sum_{i=1}^{M} \frac{1}{2} \log \left( 1 + \frac{P_i}{N_i} \right), \tag{3.33}
\]

\[
C_{\text{Partial}} = \max_{\sum_i P_i = P} \frac{1}{M} \sum_{j=1}^{M} \left[ \frac{1}{2} \log \left( 1 + \frac{P_j}{N_j + N_s} \right) + \sum_{i=1, i \neq j}^{M} \frac{1}{2} \log \left( 1 + \frac{P_i}{N_i} \right) \right]. \tag{3.34}
\]

Applying the values of our example and the optimal power distribution gives us

\[
C_{\text{Perfect}} = \sum_{i=1}^{M} \frac{1}{2} \log \left( 1 + \frac{P/M}{1} \right), \tag{3.35}
\]

\[
C_{\text{Partial}} = \frac{1}{M} \sum_{j=1}^{M} \left[ \frac{1}{2} \log \left( 1 + \frac{P/M}{1 + N_s} \right) + \sum_{i=1, i \neq j}^{M} \frac{1}{2} \log \left( 1 + \frac{P/M}{1} \right) \right]
\]
\[
= \frac{1}{2} \log \left( 1 + \frac{P/M}{1 + N_s} \right) + \sum_{i=1}^{M-1} \frac{1}{2} \log \left( 1 + \frac{P/M}{1} \right).
\] (3.36)

Taking the difference between (3.35) and (3.36) will give us the gap in capacity between the Perfect and Partial CSIR curves as a function of the total power \(P\), denoted \(g(P)\):

\[
g(P) = \frac{1}{2} \log \left( 1 + \frac{P/M}{1} \right) - \frac{1}{2} \log \left( 1 + \frac{P/M}{1 + N_s} \right).
\] (3.37)

Taking the derivative with respect to \(P\) gives us

\[
\frac{\partial g(P)}{\partial P} = \frac{1}{2 \ln 2} \left( \frac{1}{M + P} - \frac{1}{2 M(1 + N_s) + P} \right) = \frac{1}{2 \ln 2} \left[ \frac{MN_s}{(M + P)(M(1 + N_s) + P)} \right].
\] (3.38)

Since all of the parameters are nonnegative, we see that \(\frac{\partial g(P)}{\partial P} \geq 0\). Also, \(\frac{\partial g(P)}{\partial P} \to 0\) as \(P \to \infty\). Therefore, the gap in capacity between the Perfect and Partial CSIR curves can only grow or remain constant. The gap never decreases. From the analysis of (3.38), we see that the gap increases more quickly for lower \(P\) and increases very slowly for large \(P\).

Our intuition from Figure 3.4 agrees with this analysis. For increasing transmission power in the presence of a larger interference power, more gain is to be had from CSIR. However, for sufficiently large transmission power in the presence of a lower interference power, the gain from CSIR becomes approximately constant for increasing transmission power. In particular, the analysis from (3.37) and (3.38) quantifies this behavior for the gap between the Perfect and Partial CSIR curves.

A plot of the gap size between the Perfect and No CSIR curves in Figure 3.3d along with the gap size between the Perfect and Partial CSIR curves as a function of \(P/M\) is shown in Figure 3.5 in which we see the behavior described above.

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Figure 3.5. The difference in channel capacity between Perfect and No CSIR, Partial and No CSIR, and Perfect and Partial CSIR, respectively found on the plot from top to bottom. The difference is plotted over a larger range of SNRs to demonstrate the logistical growth.
CHAPTER 4

PRACTICAL APPROACHES

In this chapter, we wish to explore the performance of a practical system that approaches the channel capacity. The block diagrams of the transmitter, receiver, and transmitter for the interference are given in Figure 2.7 and Figure 2.8.

Assuming synchronization with ideal DACs, ADCs, and LPFs, we will focus on the baseband equivalent of the system. This allows us to keep the model completely in discrete time, facilitating computer simulations. Nonetheless, we must account for effects of discretization on continuous-time processes, which we will describe below. Essentially, we need to make sure that we are scaling the noise and interference samples correctly to correspond with properties of the continuous-time processes.

4.1 Details on Implementation

Here we will give more details on the implementation of the blocks for the baseband equivalent system in Figure 2.9. We will proceed in the order in which the blocks appear, beginning with the transmitter chain and ending with the receiver chain. Note that most blocks in the transmitter have an “inverse” of sorts in the receiver, such as the LDPC encoder and decoder. We will describe the related blocks in the transmitter and receiver within the same section to facilitate understanding. Also, note that we will skip the parallel-to-serial and serial-to-parallel blocks because they do not require thorough explanation.
4.1.1 LDPC Code

We begin with the details of the LDPC code. We utilize parity check matrices from the Digital Video Broadcasting standard DVB-S.2, which, in conjunction with a BCH code, can achieve a packet error rate below $10^{-7}$ about 0.7 dB to 1 dB from the Shannon limit \[26, 27\]. In this thesis, we utilize a rate 1/2 code and a rate 2/5 code with parity-check matrix dimensions of $(32400 \times 64800)$ and $(38880 \times 64800)$, respectively. These matrices correspond to irregular LDPC codes with a lower triangular matrix in the second half of the matrix.

For decoding, the system requires soft inputs from a bit-wise log-likelihood ratio, which will be discussed more in Section 4.1.5. Belief propagation is used with a maximum of 50 iterations before a hard decision is output regarding the information bits.

4.1.2 Look-Up Table

The look-up table (LUT) maps bits into symbols based on a symbol constellation. For our system, we utilize $\tilde{M}$-ary Quadrature Amplitude Modulation ($\tilde{M}$-QAM) with a Gray coded constellation. Consequently, $\log_2(\tilde{M})$ bits are mapped into a complex symbol $a$. We let $\tilde{M}$ be 4 or 16 based upon our desired bit-rate. The Gray coded constellations for 4-QAM and 16-QAM are shown in Figure 4.1 with both binary and decimal values.

\footnote{These matrices are generated in MATLAB via the function $h = \text{dvbs21dpc}(r)$, where $r$ is the rate of the code.}

\footnote{We wish to distinguish between $M$ and $\tilde{M}$. $M$ denotes the number of OFDM subcarriers in the system. $\tilde{M}$ will denote the cardinality of the symbol constellation.}
4.1.3 Fast Fourier Transform

For the transmitter, after the symbols have been produced and shifted into $M$-parallel symbols corresponding to the $M$ sub-carriers of the system, an $M$-point Inverse Fast Fourier Transform (IFFT) is performed. The IFFT effectively modulates the symbols onto orthonormal basis functions given by

$$\phi_k(t) = \begin{cases} \frac{1}{\sqrt{M}} e^{j2\pi kt/T_M}, & 0 \leq t \leq T_M \\ 0, & \text{otherwise} \end{cases}, \quad (4.1)$$

where $k = 0, 1, \ldots, M - 1$, and $T_M$ is the OFDM symbol time.

Due to sampling considerations and the effect of sampling on the variance of AWGN (see Section [4.1.4]), we must properly scale our IFFT and later the Fast Fourier Transform (FFT) in the receiver. In particular, since the IFFT and FFT
occur in discrete time, we must note that the projections

\[ \int_0^{T_M} \phi_k(t)\phi_j^*(t) \, dt = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases} \tag{4.2} \]

will become

\[ \sum_{n=0}^{M-1} \phi_k(nT)\phi_j^*(nT) = \begin{cases} \frac{1}{T}, & k = j \\ 0, & k \neq j \end{cases} \tag{4.3} \]

due to the approximation

\[ \int_0^{T_M} \phi_k(t)\phi_j^*(t) \, dt \approx T \sum_{n=0}^{M-1} \phi_k(nT)\phi_j^*(nT) \tag{4.4} \]

for small \( T \) \[10, \text{Sec. 6.5} \]. Note that for an OFDM system, the sampling time \((T)\) is related to the OFDM symbol time \((T_M)\) and the number of sub-carriers \((M)\) by \( T = \frac{T_M}{M} \).

Consequently, in calculating the received symbol \( r_k \) in Figure 2.7b as a projection of the received signal onto the basis functions, we see

\[ r_k = \int_0^{T_M} \tilde{y}(t)\phi_k^*(t) \, dt \tag{4.5} \]

\[ = \frac{1}{\sqrt{T_M}} \int_0^{T_M} \tilde{y}(t)e^{-j2\pi kt/T_M} \, dt \tag{4.6} \]

\[ \approx \frac{T}{\sqrt{T_M}} \sum_{n=0}^{M-1} \tilde{y}(nT)e^{-j2\pi kn/M}, \tag{4.7} \]

where \( \tilde{y}(t) \) is the complex-equivalent of the real-valued bandpass signal \( y(t) = \Re[\tilde{y}(t)e^{j2\pi ft}] \) \[3\].

Thus, \( r_k \) can be found as a discrete Fourier transform (DFT) of \( \tilde{y}(t) \) scaled by \( \frac{T}{\sqrt{T_M}} = \sqrt{\frac{T}{M}} \).

\[ ^3\Re[\cdot] \text{ denotes the real part of the argument.} \]
In a similar way for the transmitter, we want to modulate \( M \) symbols \([a_0, \ldots, a_{M-1}]\) by the basis functions to form a baseband OFDM symbol in time. Let \( \tilde{x}(t) \) be the baseband complex equivalent signal of \( x(t) \). Then modulation by the basis functions gives

\[
\tilde{x}(t) = \sum_{k=0}^{M-1} a_k \phi_k(t) \quad (4.8)
\]

\[
= \frac{1}{\sqrt{T_M}} \sum_{k=0}^{M-1} a_k e^{j2\pi kt/T_M}. \quad (4.9)
\]

Sampling \( \tilde{x}(t) \) with sample time \( T \) gives

\[
\tilde{x}(nT) = \frac{1}{\sqrt{T_M}} \sum_{k=0}^{M-1} a_k e^{j2\pi kn/M}, \quad (4.10)
\]

which is an inverse DFT\(^4\) scaled by \( \sqrt{\frac{M}{T}} \).

Scaling the IFFT and FFT in this way not only provides continuity with the theory, but also enables a proper relationship between the desired signal and the noise, as we will discuss in Section 4.1.4.

### 4.1.4 Sampled AWGN

For our system, we model the noise signal \( z(t) \) in Figure 2.3 as a white Gaussian random process with power spectral density (W/Hz) given by the constant

\[
S_Z(f) = \frac{N_0}{2}. \quad (4.11)
\]

At the receiver, the noise \( z(t) \) proceeds through an ideal low pass filter, thereby band-limiting the noise to frequencies below \( 1/2T \), where \( T \) is the sampling period.

---

\(^4\)We are utilizing MATLAB’s definition of the inverse DFT given by \( x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N} \) for \( n = 0, 1, \ldots, N-1 \).
at the transmitter. Let us define \( w(t) \) as output of the filter due to \( z(t) \). Then

\[
S_W(f) = \begin{cases} 
\frac{N_0}{2}, & |f| \leq \frac{1}{2T} \\
0, & |f| > \frac{1}{2T}.
\end{cases}
\] (4.12)

Since the power spectral density is the Fourier transform of the signal’s autocorrelation, we can determine the autocorrelation of \( w(t) \) to be

\[
R_{WW}(\tau) = \frac{N_0 \sin \left( \frac{\pi \tau}{T} \right)}{\pi \tau}
\] (4.13)

because the inverse transform of a rectangle gives a sinc pulse \([16, \text{Sec. 4.5}]\).

Since the filter is an LTI system and is consequently a linear operation, \( w(t) \) is a zero-mean jointly Gaussian random process \([16, \text{Th. 7.4.1}]\). To determine the autocorrelation matrix, we note that the discrete autocorrelation function for the sequence \( w[n] \) will be the values of the continuous autocorrelation function evaluated at the samples:

\[
R_{WW}[k] = E[w(nT)w^*((n - k)T)] = R_{WW}(kT) = \frac{N_0 \sin \pi k}{2T \pi k} = \frac{N_0}{2T} \delta[k],
\] (4.14)

where \( \delta[k] \) is the Kronecker delta function. As a result, the correlation matrix of the sequence is a diagonal matrix with the common entry \( N_0/2T \). Therefore, the samples \( w(nT) \) are uncorrelated, jointly Gaussian random variables with variance \( N_0/2T \). Since uncorrelated, jointly Gaussian random variables are independent \([24, \text{Sec. 9.3}]\), the samples \( w(nT) \) are independent, identically-distributed (iid) zero-mean Gaussian random variables \( i.e. w(nT) \sim \text{iid } \mathcal{N}(0, \frac{N_0}{2T}) \).

Next, the samples \( w(nT) \) must pass through the FFT block, which is equivalent to projecting the noise onto the orthonormal basis functions. Let us define the sequence \( v_k, k = 0, 1, \ldots, M - 1 \), to be the output of the FFT block due to the input \( w(nT) \),
\[ n = 0, 1, \ldots, M - 1. \] Since linear operations preserve Gaussianity, the terms \( v_k \) are Gaussian random variables. Additionally, since representing white Gaussian noise in terms of any orthonormal expansion results in iid random variables [16, Sec. 7.7], we can say that the sequence of \( v_k \) contains iid Gaussian random variables. The mean of \( v_k \) is clearly zero since no constant has been added. However, the variance requires some additional analysis.

Let us briefly look at the DFT as a linear operation. We begin by representing the DFT as a projection onto the orthonormal basis of our pulse shapes:

\[
v_k = \sum_{n=0}^{M-1} w(nT) e^{-j2\pi kn/M} \tag{4.15}
\]

\[
= \sqrt{T_M} \sum_{n=0}^{M-1} w(nT) \phi_k(nT). \tag{4.16}
\]

Next, we define the vector notation \( w = [w(0), w(T), w(2T), \ldots, w((M-1)T)]^T \) and \( v = [v_0, v_1, \ldots, v_{M-1}]^T \), and we define the transformation matrix

\[
A = \begin{bmatrix}
\phi_0(0) & \phi_1(0) & \cdots & \phi_{M-1}(0) \\
\phi_0(T) & \phi_1(T) & \cdots & \phi_{M-1}(T) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_0((M-1)T) & \phi_1((M-1)T) & \cdots & \phi_{M-1}((M-1)T)
\end{bmatrix}. \tag{4.17}
\]

Consequently, we can write

\[
v = \sqrt{T_M} A^T w. \tag{4.18}
\]

Since we know \( E[w] = 0 \), we can write

\[
E[v] = E[\sqrt{T_M} A^T w] = \sqrt{T_M} A^T E[w] = 0,
\]

which mathematically shows that the \( v_k \) are zero mean. With regard to variance,

\[
E[vv^t] = E[\sqrt{T_M} A^T w w^t A^* \sqrt{T_M}] \tag{4.19a}
\]
\[ = T_M A^T E[w w^t] A^* \]  
\[ = T_M A^T (I \sigma_w^2) A^* \]  
\[ = \sigma_w^2 T_M A^T A^* , \]  
where \((\cdot)^t\) denotes the Hermitian transpose, \((\cdot)^*\) denotes the conjugate, and \(\sigma_w^2\) is the variance of samples \(w(nT)\), which is equal to \(N_0 / 2T\).

Next, we must consider the value of \(A^T A^*\). Since the elements of \(A\) are the discrete orthonormal basis functions, \(A^T A^*\) will be a matrix with element \((j,k)\) equal to \(4.3\). Consequently, \(A^T A^* = \frac{1}{T} I\). Substituting this result into (4.19d) gives us

\[ E[vv^T] = \sigma_w^2 T_M \left( \frac{1}{T} I \right) \]  
\[ = T_M \frac{N_0}{2T^2} I. \]  
(4.20a)

This means that the jointly Gaussian \(v_k\) are uncorrelated, and consequently independent. Additionally, all of the \(v_k\) have a common variance of \(\sigma_w^2 = T_M \frac{N_0}{2T^2}\). Thus, we can say that the projected noise samples are iid Gaussian and \(v_k \sim \mathcal{N}(0, T_M \frac{N_0}{2T^2})\).

For our simulation, we desire the noise samples after the projection \((v_k)\) to be iid \(\mathcal{N}(0, \frac{N_0}{2})\) random variables. Consequently, we must scale our FFT by \(\frac{T}{\sqrt{T_M}} = \sqrt{T / M}\), which justifies the scaling in Section 4.1.3.

Due to the same reasons given above, the real and imaginary parts of the non-zero symbol for the interference transmitter \((n_j)\) in Figure 2.8 will be considered \(\mathcal{N}(0, \frac{N_0}{2T})\) random variables, and the IFFT will be of the form \(\frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} a_k e^{j2\pi kn/N}\) for \(n = 0, 1, \ldots, N-1\). This will ensure that the variance of the interference in the quadrature and inphase components will be \(N_0/2\) after the FFT.
4.1.5 Log-Likelihood Ratio

The LDPC decoder requires soft inputs in the form of bit-wise log-likelihood ratios (LLRs), which we will describe next. In general, we want to compare the likelihood that a bit is 0 given the received complex symbol and the likelihood that a bit is 1 given the received complex symbol.

We begin by comparing a-posteriori probabilities that a bit equals 0 or 1 given the received symbol \( r \) [16, Sec. 8.2]. Let \( B \in \{0, 1\} \) be a Bernoulli(1/2) random variable representing the bit of interest. Noting that the distribution of random received symbol \( R \) is continuous because of the continuous distribution of the noise and interference, we can write the a-posteriori probability as

\[
p_{B|R}(b|r) = \frac{p_B(b)f_{R|B}(r|b)}{f_R(r)}. \tag{4.21}
\]

Comparing the a-posteriori probabilities that \( B = 1 \) versus \( B = 0 \) gives us the hypothesis test

\[
p_{B|R}(0|r) = \frac{p_B(0)f_{R|B}(r|0)}{f_R(r)} \overset{\hat{B}=0}{\propto} \frac{p_B(1)f_{R|B}(r|1)}{f_R(r)} = p_{B|R}(1|r), \tag{4.22}
\]

where \( \hat{B} \) is the estimate of \( B \). Let us multiply both sides of the test by \( f_R(r) \) and divide by \( p_B(0) \) and \( f_{R|B}(r|1) \) to get

\[
\frac{f_{R|B}(r|0)}{f_{R|B}(r|1)} \overset{\hat{B}=0}{\propto} \frac{p_B(1)}{p_B(0)} = 1. \tag{4.23}
\]

The conditional densities \( f_{R|B}(r|b), b \in \{0, 1\}, \) are called likelihoods, and consequently, the term on the left is a likelihood ratio. If we take the natural logarithm of
both sides, we obtain a hypothesis test on the log-likelihood ratio:

\[
\text{LLR} = \ln \frac{f_R | B(r|0)}{f_R | B(r|1)} \overset{\hat{B} = 0}{\underset{\hat{B} = 1}{\gtrless}} 0. 
\] (4.24)

However, this hypothesis test makes a hard decision on the bit being 0 or 1. Instead, we can use the LLR itself as a soft decision for which we can measure the likelihood of the decision. LLRs that are more positive indicate a greater likelihood of the bit being 0, and LLRs that are more negative indicate a greater likelihood of the bit being 1. In contrast, LLRs close to 0 indicate a weak likelihood.

Next, let us express the likelihoods in terms of the transmitter symbol \(a \in A\). Let us define the set \(A_0\) as the set of constellation points that contain a 0 for the bit of interest. For example, say we are interested in the second bit of a symbol in the 16-QAM constellation given in Figure 4.1a. In this case, the set in decimal notation is \(A_0 = \{0, 1, 2, 3, 8, 9, 10, 11\}\). We define \(A_1\) in an analogous way.

By the laws of probability, we have the following equalities:

\[
f_{R|B}(r|b) = \frac{f_{R,B}(r,b)}{p_B(b)} 
\]

(4.25a)

\[
= \frac{\sum_{a \in A} f_{R,B|A}(r,b|a)p_A(a)}{\sum_{a \in A} p_{B|A}(b|a)p_A(a)} 
\]

(4.25b)

\[
= \frac{\sum_{a \in A} 1\{a \in A_b\} f_{R|A}(r|a)}{\sum_{a \in A} 1\{a \in A_b\}} 
\]

(4.25c)

\[
= \frac{\sum_{a \in A_b} f_{R|A}(r|a)}{\sum_{a \in A_b} 1} 
\]

(4.25d)

\[
= \frac{\sum_{a \in A_b} f_{R|A}(r|a)}{|A_b|}. 
\]

(4.25e)

We get (4.25b) by the law of total probability. Since \(p_A(a)\) is a uniform probability mass function, \(p_A(a)\) has a constant value over the summation and can be canceled in the fraction. Additionally, since \(b\) is a deterministic function of \(a\), the random variables \(R\) and \(B\) are conditionally independent given \(a\), and \(p_{B|A}(b|a)\) is an indicator
function of the event \( a \in A_b \). Consequently, (4.25c) results. The indicator function acts to reduce the set over which the summation occurs, which gives us (4.25d). If we define \(|A_b|\) to be the cardinality of the set \( A_b \), we have (4.25e). Note that
\[ |A_b| = \tilde{M}/2 \text{ for } b \in \{0, 1\}. \]

Consequently, we have as our likelihoods
\[ f_{R|B}(r|0) = \frac{1}{|A_0|} \sum_{a \in A_0} f_{R|A}(r|a), \quad (4.26) \]
\[ f_{R|B}(r|1) = \frac{1}{|A_1|} \sum_{a \in A_1} f_{R|A}(r|a). \quad (4.27) \]

Since we will be interested in the ratio of (4.26) and (4.27) and \(|A_0| = |A_1|\), We simply need to determine the conditional density \( f_{R|A}(r|a) \) for each case of CSIR.

For Perfect CSIR, we subtract the interference off the received symbols and are left with the symbols plus AWGN. Since the inphase and quadrature components of the noise will be independent (and identically distributed) [16, Sec. 7.7], the conditional density \( f_{R|A}(r|a) \) will a bivariate Gaussian distribution with independent components. Let \( r_R \) and \( a_R \) denote the real (inphase) part of \( r \) and \( a \), respectively, and let \( r_I \) and \( a_I \) denote the imaginary (quadrature) part of \( r \) and \( a \). In other words, \( f_{R|A}(r|a) = f_{R|R}(r_R|a_R) \cdot f_{R|I}(r_I|a_I) \) where \( f_{R|R}(r_R|a_R) \sim \mathcal{N}(a_R, \frac{N_0}{2}) \) and \( f_{R|I}(r_I|a_I) \sim \mathcal{N}(a_I, \frac{N_0}{2}) \). Thus, the LLR for Perfect CSIR is given by
\[ \text{LLR} = \ln \left[ \frac{\sum_{a \in A_0} e^{-\frac{1}{2} \left( \frac{(r_R-a_R)^2}{N_0/2} + \frac{(r_I-a_I)^2}{N_0/2} \right)} \left( \sum_{a \in A_0} e^{-\frac{1}{2} \left( \frac{(r_R-a_R)^2}{N_0/2} + \frac{(r_I-a_I)^2}{N_0/2} \right)} \right)}{\sum_{a \in A_1} e^{-\frac{1}{2} \left( \frac{(r_R-a_R)^2}{N_0/2} + \frac{(r_I-a_I)^2}{N_0/2} \right)} \left( \sum_{a \in A_1} e^{-\frac{1}{2} \left( \frac{(r_R-a_R)^2}{N_0/2} + \frac{(r_I-a_I)^2}{N_0/2} \right)} \right)} \right]. \quad (4.28) \]

For Partial CSIR, we know which sub-carrier is experiencing the additional Gaussian noise due to the interference. Consequently, for the sub-carrier with interference, the conditional densities will have a larger variance due to the interference. Specifically, we have
\[ f_{R|A}(r|a) = \frac{1}{2\pi(N_0/2 + N_0J/2)} e^{-\frac{1}{2} \left( \frac{(r_R-a_R)^2}{N_0/2+N_0J/2} + \frac{(r_I-a_I)^2}{N_0/2+N_0J/2} \right)} . \]  

For the remaining sub-carriers, the LLR reduces to (4.28) since only noise corrupts the received symbols. Hence, the LLR for Partial CSIR is given by

\[
\text{LLR} = \begin{cases} 
\ln \left[ \frac{\sum_{a \in A_0} e^{-\frac{1}{2} \left( \frac{(r_R-a_R)^2}{N_0/2+N_0J/2} + \frac{(r_I-a_I)^2}{N_0/2+N_0J/2} \right)}}{\sum_{a \in A_1} e^{-\frac{1}{2} \left( \frac{(r_R-a_R)^2}{N_0/2+N_0J/2} + \frac{(r_I-a_I)^2}{N_0/2+N_0J/2} \right)}} \right], & \text{sub-carrier with interference} \\
\ln \left[ \frac{\sum_{a \in A_0} e^{-\frac{1}{2} \left( \frac{(r_R-a_R)^2}{N_0/2} + \frac{(r_I-a_I)^2}{N_0/2} \right)}}{\sum_{a \in A_1} e^{-\frac{1}{2} \left( \frac{(r_R-a_R)^2}{N_0/2} + \frac{(r_I-a_I)^2}{N_0/2} \right)}} \right], & \text{otherwise.} 
\end{cases}
\]

(4.30)

For No CSIR, we know that in addition to the noise, interference is present. The interference is assumed to add additional zero-mean Gaussian noise with variance \( \frac{N_0J}{2} \) to a single sub-carrier, while the remaining \( M - 1 \) sub-carriers experience only the AWGN. The interference has independent inphase and quadrature components.

We note that, for optimality, we must consider all \( M \) received symbols for a single OFDM symbol jointly. This is due to the fact that the interference is only present on one sub-carrier for a given OFDM symbol. Let us define the notation \( a = [a_0, a_1, \ldots, a_{M-1}]^T \) and \( r = [r_0, r_1, \ldots, r_{M-1}]^T \). Additionally, let the random variable \( B_i \) represent the \( i \)th bit contained in the vector of \( \tilde{M} \)-QAM symbols. Note that \( M \log_2(\tilde{M}) \) bits are contained within a vector of QAM symbols.

Beginning again from the a-posteriori probability \( p_{B_i|R}(b_i|r) \) and following the same procedure as equations (4.21) through (4.24), we find the LLR for the \( i \)th bit to be
LLR = \ln \frac{f_{\mathbf{R}|\mathbf{B}_i}(\mathbf{r}|0)}{f_{\mathbf{R}|\mathbf{B}_i}(\mathbf{r}|1)}. \tag{4.31}

Next, we wish to express the likelihoods in terms of the transmitted vector of QAM symbols \( \mathbf{a} \in \mathcal{A}^M \). We use the same strategy employed previously in equations (4.25b) through (4.25e):

\[
\begin{align*}
    f_{\mathbf{R}|\mathbf{B}_i}(\mathbf{r}|b_i) &= \frac{f_{\mathbf{R}|\mathbf{B}_i}(\mathbf{r}, b_i)}{p_{B_i}(b_i)} \tag{4.32a} \\
    &= \frac{\sum_{a \in \mathcal{A}^M} f_{\mathbf{R}|\mathbf{B}_i,\mathcal{A}}(\mathbf{r}, b_i|a)p_{\mathcal{A}}(a)}{\sum_{a \in \mathcal{A}^M} p_{B_i}(b_i|a)p_{\mathcal{A}}(a)} \tag{4.32b} \\
    &= \frac{\sum_{a \in \mathcal{A}^M} 1\{a \in \mathcal{A}^M_{b_i}\} f_{\mathbf{R}|\mathcal{A}}(\mathbf{r}|a)}{\sum_{a \in \mathcal{A}^M} 1\{a \in \mathcal{A}^M_{b_i}\}} \tag{4.32c} \\
    &= \frac{\sum_{a \in \mathcal{A}^M_{b_i}} f_{\mathbf{R}|\mathcal{A}}(\mathbf{r}|a)}{\sum_{a \in \mathcal{A}^M_{b_i}} 1} \tag{4.32d} \\
    &= \frac{\sum_{a \in \mathcal{A}^M_{b_i}} f_{\mathbf{R}|\mathcal{A}}(\mathbf{r}|a)}{|\mathcal{A}^M_{b_i}|}. \tag{4.32e}
\end{align*}
\]

We define \( \mathcal{A}^M_{b_i} \) to be the Cartesian product of the constellation \( \mathcal{A} \) with itself \( M \) times, with the \( i \)th constellation being constrained to \( \mathcal{A}_0 \) or \( \mathcal{A}_1 \). Note that \( |\mathcal{A}^M_0| = |\mathcal{A}^M_1| \).

Therefore, our joint LLR is given by

\[
\text{LLR} = \ln \frac{\sum_{a \in \mathcal{A}^M_0} f_{\mathbf{R}|\mathcal{A}}(\mathbf{r}|a)}{\sum_{a \in \mathcal{A}^M_1} f_{\mathbf{R}|\mathcal{A}}(\mathbf{r}|a)}. \tag{4.33}
\]

We now look at \( f_{\mathbf{R}|\mathcal{A}}(\mathbf{r}|a) \) and note that conditioning on \( J \) results in a multivariate Gaussian distribution with a diagonal covariance matrix. In other words,

\[
f_{\mathbf{R}|\mathcal{A}}(\mathbf{r}|a) = \sum_{j=1}^{M} p_{J}(j) f_{\mathbf{R}|\mathcal{A},J}(\mathbf{r}|a, j). \tag{4.34}
\]
and

\[
\begin{align*}
\tilde{f}_{R|\mathbf{A},j}(R|\mathbf{a},j) &= \frac{1}{(2\pi)^{M/2} \det(K_j)^{1/2}} \exp\left(-\frac{1}{2} (R - \mathbf{a})^T K_j^{-1} (R - \mathbf{a}) \right), \tag{4.35}
\end{align*}
\]

where the covariance matrix

\[
K_j = \begin{bmatrix}
    N_0/2 \\
    \vdots \\
    N_0/2 + N_0J/2 \\
    \vdots \\
    N_0/2
\end{bmatrix}
\]

is a diagonal matrix with entries $N_0/2$ except for element $(j, j)$ which equals $N_0/2 + N_0J/2$. Hence, we can evaluate the joint LLR in (4.33) by calculating the quantities in (4.34) through (4.36).

However, summing over $\mathbf{a} \in \mathbf{A}_0^M$ and $\mathbf{a} \in \mathbf{A}_1^M$ is an exhaustive search with $\tilde{M}^M$ possible vectors. For the examples presented earlier in which $M = 16$ and $\tilde{M} = 16$, we would have to search over $16^{16} \approx 1.8 \times 10^{19}$ terms. Consequently, such an implementation is time-consuming and resource-consuming. We will consider a sub-optimal “marginal” LLR.

We will take a probabilistic approach and consider each sub-carrier. We know $M - 1$ sub-carriers will only experience noise with one sub-carrier experiencing some additional interference. Therefore, we write

\[
\begin{align*}
f_{R|\mathbf{A}}(R|\mathbf{a}) &= \frac{1}{M} \frac{1}{2\pi (N_0/2 + N_0J/2)} e^{-\frac{1}{2} \left( \frac{r_R - a_R)^2}{N_0/2 + N_0J/2} + \frac{(r_I - a_I)^2}{N_0/2} \right)} \\
&+ \frac{M - 1}{M} \frac{1}{2\pi N_0/2} e^{-\frac{1}{2} \left( \frac{(r_R - a_R)^2}{N_0/2} + \frac{(r_I - a_I)^2}{N_0/2} \right)}. \tag{4.37}
\end{align*}
\]
Consequently, our sub-optimal LLR for No CSIR is given by

\[
\text{LLR} = \ln \left[ \frac{\sum_{a \in A_0} \frac{1}{N_0/2 + N_{i,j}/2} e^{-\frac{1}{2} \left( \frac{(r_R - a_R)^2}{N_0/2} + \frac{(r_I - a_I)^2}{N_0/2} \right)} + \frac{M-1}{N_0/2} e^{-\frac{1}{2} \left( \frac{(r_R - a_R)^2}{N_0/2} + \frac{(r_I - a_I)^2}{N_0/2} \right)}}{\sum_{a \in A_1} \frac{1}{N_0/2 + N_{i,j}/2} e^{-\frac{1}{2} \left( \frac{(r_R - a_R)^2}{N_0/2} + \frac{(r_I - a_I)^2}{N_0/2} \right)} + \frac{M-1}{N_0/2} e^{-\frac{1}{2} \left( \frac{(r_R - a_R)^2}{N_0/2} + \frac{(r_I - a_I)^2}{N_0/2} \right)}} \right].
\]

(4.38)

With all of the blocks described in detail, we now consider our simulations.

4.2 Simulation Results

Here we will describe simulations of the system in Figure 2.9. To match the conditions of our earlier simulation, we set \( M = 16 \) sub-carriers and \( N_i = 1 \) watt for all sub-carriers. We will vary the code rate \( r \), QAM constellation size \( \tilde{M} \), the interference power \( N_s \), and the transmission power \( P \).

For a given set of system parameters, we randomly generate data bits to send through the transmitter chain, across the channel, and through the receiver chain. Our goal is to find the “signal-to-noise ratio per information bit” \( E_b/N_0 \) that would result in a bit-error rate (BER) of around \( 1 \times 10^{-6} \). To make our results statistically significant, we desire around 10,000 errors to be observed for the calculation of the BER. A graphical representation of this process is in Figure 4.2.

For the sake of space, we will only provide the intermediate steps for one example. Let us look at the case of Perfect CSIR with code rate \( r = 2/5 \) and QAM constellation size \( \tilde{M} = 4 \). Note that for Perfect CSIR the interference has no effect on the system, so \( N_s \) is not a parameter.

We begin by simulating the system to obtain some “coarse” information regarding the general shape of the BER versus \( E_b/N_0 \) curve. We select a low \( E_b/N_0 \) and simulate the system until at least 500 errors have been observed. The BER is calculated for that particular \( E_b/N_0 \). If the BER is greater than our desired BER of \( 1 \times 10^{-6} \), we
Figure 4.2. The simulation process to find the $E_b/N_0$ necessary for a BER of $10^{-6}$.

add 0.05 dB to $E_b/N_0$ and repeat. When the simulated BER is less than or equal to our desired BER of $1 \times 10^{-6}$, we stop. The result is the red dot-dashed line in Figure 4.3.

With the “coarse” bit-error rates, we focus on the $E_b/N_0$ near the $10^{-6}$ BER. This time, we simulate the system until about 10,000 errors have been observed. We repeat the process until the BER is acceptably close to $10^{-6}$. The result is the black dashed line in Figure 4.3.

For the sake of comparison, we offer the probability of a bit error for the uncoded $\tilde{M}$-QAM system which can be computed as [10, Exercise 6.5]

$$P_b \approx \frac{4}{\log_2 \tilde{M}} \left( \frac{\sqrt{\tilde{M} - 1}}{\sqrt{\tilde{M}}} \right) Q \left( \sqrt{\frac{3 \log_2 (\tilde{M}) E_b}{\tilde{M} - 1 N_0}} \right) \ldots$$

$$- \frac{4}{\log_2 \tilde{M}} \left( \frac{\tilde{M} - 2 \sqrt{\tilde{M} - 1}}{M} \right) \left[ Q \left( \sqrt{\frac{3 \log_2 (\tilde{M}) E_b}{\tilde{M} - 1 N_0}} \right) \right]^2,$$ (4.39)

where the Q-function is defined as

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} \, dt.$$ (4.40)

This result is the solid blue line in Figure 4.3.
### Figure 4.3

The simulated bit-error rate as a function of $E_b/N_0$. The red dot-dashed line corresponds to the "coarse" simulation with at least 500 errors observed per BER. The black dashed line shows BERs for which about 10,000 errors were observed. The solid blue line is the probability of bit error for the uncoded system.

Additionally, we offer the Shannon limit for comparison [16, Sec. 7.9]. For the case of Perfect CSIR, the sum capacity for a two-dimensional channel use is given by $C = M \log_2(1 + SNR)$. The information bit rate of our system per channel use (an OFDM symbol) is $R_b = Mr \log_2(\tilde{M})$. We derive this in detail in Chapter 5. For the Shannon limit, we wish to determine the $SNR^*$ necessary to achieve the information bit rate $R_b$ with an optimal system:

$$R_b = Mr \log_2(\tilde{M}) = M \log_2(1 + SNR^*) \quad (4.41)$$

$$r \log_2(\tilde{M}) = \log_2(1 + SNR^*) \quad (4.42)$$

$$2^{r \log_2(M)} - 1 = SNR^*. \quad (4.43)$$
Since our system will perform below capacity, we know the system should always have an SNR greater than SNR*. Hence, SNR* is the Shannon limit. To apply this concept to a BER curve, we need to express SNR in terms of $E_b/N_0$. We explain this relationship in detail in Chapter 5. Here we will simply say that for our system $\text{SNR} = r \log_2(M) \cdot E_b/N_0$. Thus, the Shannon limit can be expressed as

$$\text{SNR} > \text{SNR}^*$$ (4.44a)

$$\frac{E_b}{N_0} > \frac{\text{SNR}^*}{r \log_2(M)}$$ (4.44b)

$$\frac{E_b}{N_0} > \frac{2^{r \log_2(M)} - 1}{r \log_2(M)}.$$ (4.44c)

From Figure 4.3, we see that our coded system is about 0.8 dB from capacity at a BER of $10^{-6}$.

Our results for the cases of Perfect CSIR, Partial CSIR, and No CSIR are shown in Tables 4.1, 4.2, and 4.3, respectively. For the sake of space, we will focus only on the $E_b/N_0$ that achieved a BER of approximately $10^{-6}$.

**TABLE 4.1**

SIMULATION RESULTS FOR THE CASE OF PERFECT CSIR

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\tilde{M}$</th>
<th>$E_b/N_0$ (dB)</th>
<th>BER</th>
<th>Number of errors observed</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>4</td>
<td>0.910</td>
<td>$1.35 \times 10^{-6}$</td>
<td>17,372</td>
</tr>
<tr>
<td>1/2</td>
<td>16</td>
<td>3.225</td>
<td>$1.20 \times 10^{-6}$</td>
<td>16,759</td>
</tr>
<tr>
<td>2/5</td>
<td>4</td>
<td>0.494</td>
<td>$1.21 \times 10^{-6}$</td>
<td>12,505</td>
</tr>
<tr>
<td>2/5</td>
<td>16</td>
<td>2.484</td>
<td>$1.38 \times 10^{-6}$</td>
<td>15,797</td>
</tr>
</tbody>
</table>
TABLE 4.2

SIMULATION RESULTS FOR THE CASE OF PARTIAL CSIR

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\bar{M}$</th>
<th>$N_s$</th>
<th>$E_b/N_0$ (dB)</th>
<th>BER</th>
<th>Number of errors observed</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>4</td>
<td>1</td>
<td>1.065</td>
<td>$1.80 \times 10^{-6}$</td>
<td>17,982</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td></td>
<td>1.240</td>
<td>$1.37 \times 10^{-6}$</td>
<td>13,722</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td></td>
<td>1.290</td>
<td>$1.56 \times 10^{-6}$</td>
<td>15,908</td>
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<tr>
<td></td>
<td>16</td>
<td></td>
<td>1.315</td>
<td>$1.10 \times 10^{-6}$</td>
<td>11,021</td>
</tr>
<tr>
<td>1/2</td>
<td>16</td>
<td>1</td>
<td>3.450</td>
<td>$1.12 \times 10^{-6}$</td>
<td>11,180</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td></td>
<td>3.615</td>
<td>$1.19 \times 10^{-6}$</td>
<td>11,900</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td></td>
<td>3.680</td>
<td>$1.44 \times 10^{-6}$</td>
<td>14,400</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td></td>
<td>3.720</td>
<td>$1.24 \times 10^{-6}$</td>
<td>12,440</td>
</tr>
<tr>
<td>2/5</td>
<td>4</td>
<td>1</td>
<td>0.654</td>
<td>$9.62 \times 10^{-7}$</td>
<td>9,623</td>
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<tr>
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<td>12,833</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td></td>
<td>0.839</td>
<td>$1.38 \times 10^{-6}$</td>
<td>13,784</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td></td>
<td>0.859</td>
<td>$1.18 \times 10^{-6}$</td>
<td>11,793</td>
</tr>
<tr>
<td>2/5</td>
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<td>1</td>
<td>2.644</td>
<td>$1.08 \times 10^{-6}$</td>
<td>10,831</td>
</tr>
<tr>
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<td></td>
<td>2.819</td>
<td>$1.16 \times 10^{-6}$</td>
<td>11,588</td>
</tr>
<tr>
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<td>2.869</td>
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<td>16,030</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td></td>
<td>2.894</td>
<td>$1.24 \times 10^{-6}$</td>
<td>12,406</td>
</tr>
</tbody>
</table>
### TABLE 4.3

SIMULATION RESULTS FOR THE CASE OF NO CSIR

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\tilde{M}$</th>
<th>$N_s$</th>
<th>$E_b/N_0$ (dB)</th>
<th>BER</th>
<th>Number of errors observed</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/2$</td>
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<td>1</td>
<td>1.160</td>
<td>$1.14 \times 10^{-6}$</td>
<td>15,754</td>
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<td>1.600</td>
<td>$1.60 \times 10^{-6}$</td>
<td>16,066</td>
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<tr>
<td></td>
<td>10</td>
<td>1</td>
<td>1.690</td>
<td>$1.83 \times 10^{-6}$</td>
<td>12,893</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>1</td>
<td>1.690</td>
<td>$1.44 \times 10^{-6}$</td>
<td>14,358</td>
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<tr>
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<td>1</td>
<td>3.460</td>
<td>$1.48 \times 10^{-6}$</td>
<td>22,895</td>
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<td>5</td>
<td>1</td>
<td>3.940</td>
<td>$1.04 \times 10^{-6}$</td>
<td>10,395</td>
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<tr>
<td></td>
<td>10</td>
<td>1</td>
<td>4.150</td>
<td>$1.22 \times 10^{-6}$</td>
<td>14,859</td>
</tr>
<tr>
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<td>1</td>
<td>4.210</td>
<td>$1.41 \times 10^{-6}$</td>
<td>15,596</td>
</tr>
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<td>0.734</td>
<td>$1.32 \times 10^{-6}$</td>
<td>13,169</td>
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<td>1.149</td>
<td>$1.41 \times 10^{-6}$</td>
<td>14,087</td>
</tr>
<tr>
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<td>1</td>
<td>1.219</td>
<td>$1.33 \times 10^{-6}$</td>
<td>10,140</td>
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<tr>
<td></td>
<td>16</td>
<td>1</td>
<td>1.199</td>
<td>$1.54 \times 10^{-6}$</td>
<td>10,080</td>
</tr>
<tr>
<td>$2/5$</td>
<td>16</td>
<td>1</td>
<td>2.719</td>
<td>$1.17 \times 10^{-6}$</td>
<td>19,294</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>1</td>
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<td>$1.54 \times 10^{-6}$</td>
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<td></td>
<td>10</td>
<td>1</td>
<td>3.309</td>
<td>$1.41 \times 10^{-6}$</td>
<td>10,065</td>
</tr>
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<td></td>
<td>16</td>
<td>1</td>
<td>3.339</td>
<td>$1.51 \times 10^{-6}$</td>
<td>11,199</td>
</tr>
</tbody>
</table>

#### 4.3 Analysis

For equivalent parametric settings, Perfect CSIR achieves a BER around $1 \times 10^{-6}$ at lower $E_b/N_0$ than Partial CSIR. Similarly, Partial CSIR achieves the desired BER at lower $E_b/N_0$ than does No CSIR. This trend agrees with our intuition and with the expected behavior that we derived earlier from theory.
In general, with all else equal, a system with $\tilde{M} = 4$ should require a lower $E_b/N_0$ to achieve a BER than a system with $\tilde{M} = 16$. We know this from an analysis of (4.39). Our results agree with this qualitative rule.

Additionally, with all else equal, a system with code rate $r = 2/5$ generally requires a lower $E_b/N_0$ to achieve our desired BER than a system with $r = 1/2$. The reason is that the system with $r = 2/5$ will have more parity bits for given $E_b/N_0$ than the system with $r = 1/2$. Since the additional parity bits lower the BER, the system with $r = 2/5$ will, in general, have a lower BER than the system with $r = 1/2$ for a given $E_b/N_0$. Our results again conform to this idea.

Furthermore, from the results for Partial and No CSIR, we see that for a given code rate $r$ and QAM constellation size $\tilde{M}$, the $E_b/N_0$ required for a BER around $1 \times 10^{-6}$ increases with increasing interference power $N_s$. These results agree with our intuition. One exception in our results is the case of No CSIR with $r = 2/5$ and $\tilde{M} = 4$. The $E_b/N_0$ for $N_s = 10$ is greater than that for $N_s = 16$. However, if we look at the corresponding BERs, we see that if we were to reduce the BER for $N_s = 16$ to that for $N_s = 10$, we would have to increase $E_b/N_0$. Similar considerations would have to be made for $N_s = 10$ and $N_s = 16$ in the case of No CSIR with $r = 1/2$ and $\tilde{M} = 4$, but here we would argue for reducing the BER to $10^{-6}$ for both with the idea that we would need a higher $E_b/N_0$ to reduce the BER for $N_s = 16$. Evidently, for a 4-QAM system with high interference power and No CSIR, the required $E_b/N_0$ is about the same.

After all of the considerations given above, the results appear reasonable and agree with our general expectations.
CHAPTER 5
COMPARATIVE ASSESSMENT

The results in Chapters 3 and 4 are obviously related. Both sets of results proceed from two different abstractions of the same system. In this chapter, we wish to show the connection between the results and use this connection to compare the performance of the practical system to the fundamental limits derived from information theory.

5.1 Relationship Between Results

In information theory, the channel capacity is defined for an arbitrarily low probability of error. In practice, we approximate the condition of arbitrarily low probability of error with a low bit-error rate. For this thesis, we have selected a BER of $1 \times 10^{-6}$ to satisfy this condition, though often a BER of $10^{-5}$ or $10^{-7}$ is used.

A critical relationship between the results of Chapters 3 and 4 comes from the relationship between the SNR $P/N$ and the so-called “SNR per bit” $E_{b}/N_{0}$. First, we want to relate the average energy per information bit ($E_{b}$) to the total transmit power ($P$). From dimensional analysis, we see that we require the data rate (information bits/sec) of the practical system to determine the total transmit power per channel use. In other words, $P = E_{b}R_{d}$, where $R_{d}$ is the data rate in information bits per second of the practical system. Similarly, to relate the noise power spectral density ($N_{0}$) to the total noise power ($N$), we need the bandwidth of the practical system. In other words, $N = N_{0}B$, where $B$ is the bandwidth in hertz of the practical system.
Therefore, the relationship between the two SNRs is given by

$$\frac{P}{N} = \frac{E_b R_d}{N_0 B}.$$  \hspace{1cm} (5.1)

Since one channel use in Figure 2.5 corresponds to one OFDM symbol in Figure 2.7, we ultimately need the number of data bits transmitted in one OFDM symbol, knowing that the duration of one OFDM symbol is $T_M$ seconds. The number of data bits per OFDM symbol is the code rate ($r$ data bits / encoded bits), multiplied by the number of bits per QAM symbol ($\log_2(\tilde{M})$), multiplied by the number of sub-carriers $M$. Thus, $R_d = r \log_2(\tilde{M}) \cdot M/T_M$ bits per second.

Next, we consider the bandwidth of an OFDM symbol. From equation (4.1), we see that the sub-carrier spacing is $1/T_M$. With $M$ sub-carriers in an OFDM symbol, the passband bandwidth of an OFDM symbol is given by $B = M/T_M$. Consequently,

$$\frac{R_d}{B} = \frac{r \log_2(\tilde{M}) \cdot M/T_M}{M/T_M} = r \log_2(\tilde{M}).$$  \hspace{1cm} (5.2)

Combining equations (5.1) and (5.2) gives us

$$\frac{P}{N} = \frac{E_b}{N_0} r \log_2(\tilde{M}).$$  \hspace{1cm} (5.3)

One last step is to note that in our simulations $P_i = P/M$ and $N_i = N/M$. Therefore, $P/N = P_i M/N_i M = P_i/N_i$. This is an important remark because it allows us to plot the results of Tables 4.1, 4.2, and 4.3 onto Figure 3.3 as a basis for comparison. The x-axis of Figure 3.3 is equal to $P_i/N_i$ in dB and the y-axis measures the number of information bits per channel use (or OFDM symbol) per real dimension (1D). Consequently, we can plot the results of Chapter 4 onto Figure 3.3 by using (5.3) to determine x-coordinates. For the y-coordinates, we must take into account
that the practical system transmits over two dimensions via complex-valued symbols. Consequently, we must use $\frac{1}{2} M r \log_2(\tilde{M})$ to determine the bits per OFDM symbol per real dimension for the y-coordinates.

5.2 Results

We plot the results from Tables 4.1, 4.2, and 4.3 with the results of Figure 3.3. For ease of comparison, we plot the cases of Perfect, Partial, and No CSIR separately. The results for Perfect CSIR, for which the parameter $N_s$ has no effect, are found in Figure 5.1. The results for Partial CSIR are found in Figure 5.2. Finally, the results for No CSIR are found in Figure 5.3.

Figure 5.1. The data rates and SNRs for our practical system to achieve a BER of $10^{-6}$ compared with the channel capacity for the case of Perfect CSIR.
Figure 5.2. The data rates and SNRs for our practical system to achieve a BER of $10^{-6}$ compared with the channel capacity for the case of Partial CSIR.

5.3 Analysis

For a given data rate, the practical system requires higher SNRs than that of channel capacity to achieve a BER of $10^{-6}$. This meets our expectations, of course, since channel capacity demonstrates the fundamental limits of performance.
To check our results further, we would like to consider the gap between the $E_b/N_0$ achieving a BER of $10^{-6}$ and the Shannon limit in Figure 4.3. The gap is a little larger than 0.8 dB. This means that the system performs about 0.8 dB away from capacity for a rate $2/5$ code with 4-QAM. In Figure 5.1, the lowest data point for the practical system corresponds to these parameters. We can see that the gap between this point and the capacity curve is about 0.8 dB, as well. Hence, the results in Figure 4.3 and Figure 5.1 agree as they should.

Figure 5.3. The data rates and SNRs for our practical system to achieve a BER of $10^{-6}$ compared with the approximate channel capacity for the case of No CSIR.
Ultimately, we see from our results that a practical implementation of a system can approach the performance of the capacity curves. A gap ranging from fractions of a decibel to 2 dB still exists between capacity and the performance of the practical system. Nonetheless, the relative gains achieved by CSIR for the fundamental limits appear to be close for the practical system, as well.

When we also account for synchronization, the data rate of the system will lower slightly since overhead in the form of training symbols must be sent before the information. Consequently, we can view the results of the practical system as another “upper bound” of sorts for the performance of a real system.
6.1 Summary of Results

With the growing prevalence of OFDM in wireless transmission for high-throughput systems and the increased usage of spectrum, the case of interference-limited communication with OFDM will undoubtedly rise in occurrence. As a contribution to an investigation into the effects of interference on an OFDM system, this thesis analyzed the performance of an OFDM system in the presence of a frequency-hopping interferer.

To do this, we began with high level abstractions of the system in Chapter 2. The abstraction at the highest level allowed us to consider the fundamental limits of system performance in Chapter 3. Specifically, we derived the Shannon channel capacity for the system in the cases of Perfect and Partial CSIR and derived the optimal transmit power distribution for the case of Partial CSIR in Appendix A. Additionally, for the case of No CSIR, we explored computing the mutual information between two random vectors with distributions that are Gaussian mixtures. We demonstrated the insufficiency of the analytical approach with the Taylor Series expansion and justified the use of Monte Carlo methods instead. With the Monte Carlo method, we approximated the channel capacity for the case of No CSIR by computing the mutual information between the input and output vectors of the system with the assumption of a Gaussian input and evenly distributed power at the transmitter.

From the derived and approximated channel capacities in Chapter 3, we calculated the gains in SNR from CSIR. The results indicate a trade-off between the gain...
in SNR and the computational complexity of estimating the characteristics of the interference signal. More explicitly, determining the realization of the interference signal garners the most gain but is the most computationally complex. On the other hand, determining the location of the interference among the sub-carriers is not nearly as computationally complex but achieves only a fraction of the potential gain. We found that the gain in SNR for Partial CSIR as a fraction of the gain for Perfect CSIR reaches a maximum value for moderate interference powers. This suggests a strategy for which we estimate the location of the interference among the sub-carriers for moderate interference powers and treat the interference as noise at low and very high interference powers.

In order to determine if these theoretical insights would be applicable to a practical implementation of the system, we devoted Chapter 4 to our more practical abstraction of an OFDM system in Chapter 2. We computed the BER of the system as a function of $E_b/N_0$ and found the $E_b/N_0$ for which the BER was approximately $10^{-6}$, which was intended to correspond to an “arbitrarily low probability of error” that is required by the definition of Shannon capacity.

Finally, we devoted Chapter 5 to compare the BER results with the channel capacities found earlier. We determined the relationship between $E_b/N_0$ and SNR and the relationship between bits per channel use and the system parameters. These relationships enabled both sets of results to be plotted together, which showed that the practical system, despite performing 0.8 to 2 dB below capacity, could approach the channel capacity. Additionally, the comparison indicated that the gains from CSIR suggested by theory would be approximately true for an implemented system. Thus, the previously suggested strategy for trading-off computational complexity and SNR gain depending on the interference power could benefit a practical system.
6.2 Future Extensions

This thesis offers the possibility of several extensions. We will briefly highlight a few that we think would be of particular benefit.

First, our framework requires flat fading. In order to make the results applicable to a wider class of scenarios, we should consider fading in the high level abstractions of our system. A further extension in this regard would be to consider fast fading, as well.

Second, the frequency-hops of the interference are assumed to be uniform across the sub-carriers. A particularly useful extension would be to consider an interferer whose hops are determined by a Markov chain (of varying orders). The additional structure of the interference suggests the potential for more gain, but achieving such gain may require feedback to the transmitter.

Third, our estimation of channel capacity for the case of No CSIR involved several assumptions and a simulated result. It would be beneficial to determine the conditions under which Huber’s second order Taylor Series approximation would be appropriate. Additionally, determining some analytical means to calculate the capacity for the case of No CSIR would be very beneficial. Such a result could verify our results and assumptions regarding the optimal input distribution and power distribution.

Finally, our high level abstraction of the problem assumes away issues regarding synchronization. A challenging, but valuable, extension would be to add timing, phase, and frequency synchronization blocks to the practical system. Consideration of these issues may preclude simple simulation of the system. Experiments with software defined radios would need to be utilized. This set-up would also force us to address the issue of the interference being synchronous or asynchronous with the desired system.
APPENDIX A

PROOF FOR OPTIMAL TRANSMITTER POWER DISTRIBUTION IN THE CASE OF PARTIAL CSIR

We claim that for our system of interest, the capacity for the case of Partial CSIR is given by

\[
C = \max_{\sum_i p_i \leq P} \frac{1}{M} \sum_{j=1}^{M} \left[ \frac{1}{2} \log \left( 1 + \frac{P_j}{N_j + N_s} \right) + \sum_{i=1 \atop i \neq j}^{M} \frac{1}{2} \log \left( 1 + \frac{P_i}{N_i} \right) \right],
\]

where we assume that \( p_J(j) \) is a discrete uniform distribution over \( \{1, \ldots, M\} \). In short, the optimal power distribution is found via water-filling. The proof follows in which the Kuhn-Tucker conditions (given in [28, Th. 14.18]) are shown to be satisfied with this strategy. Additionally, we show that since the feasible set is convex and the objective function is concave, the solution is globally optimum.

A.1 Kuhn-Tucker Conditions

Let us define \( \mathbf{P} = [P_1, \ldots, P_M]^T \) and similarly define \( \lambda \) to be a vector of \( \lambda_i \) corresponding to each constraint of the optimization problem.

Our objective function is given by

\[
f(\mathbf{P}) = \frac{1}{M} \sum_{j=1}^{M} \left[ \frac{1}{2} \log \left( 1 + \frac{P_j}{N_j + N_s} \right) + \sum_{i=1 \atop i \neq j}^{M} \frac{1}{2} \log \left( 1 + \frac{P_i}{N_i} \right) \right],
\]
with the constraints

\[
\sum_{i=1}^{M} P_i \leq P \quad \text{(A.3a)}
\]

\[
P_i \geq 0 \quad \forall \ i, \quad \text{(A.3b)}
\]

where \( P \) is the total power available to the transmitter. The constraint given by (A.3a) can be rewritten as \( P - \sum_i P_i \geq 0 \). This will help us to write the problem in a standard form.

By subtracting the constraints from the objective with Lagrangian multipliers, we have the following Lagrangian equation:

\[
\mathcal{L}(P, \lambda) = \frac{1}{M} \sum_{j=1}^{M} \left[ \frac{1}{2} \log \left( 1 + \frac{P_j}{N_j + N_s} \right) + \sum_{i=1}^{M} \left( \frac{1}{2} \log \left( 1 + \frac{P_i}{N_i} \right) \right) \right] - \lambda_0 \left( P - \sum_{i=1}^{M} P_i \right) - \sum_{i=1}^{M} \lambda_i P_i,
\]

(A.4)

where the constraints are subtracted because this is a maximization problem and deviations from the constraints will decrease our function.

Next, let us write out the Kuhn-Tucker conditions according to [28].

**Theorem 1.** (Kuhn-Tucker Conditions) For an objective function \( f(x) \) subject to the constraints \( g(x) \geq 0 \), the sufficient conditions for a solution \( x^* \) to be a local minimizer are the following:

1. \( x^* \) satisfies \( g(x^*) \geq 0 \),
2. \( x^* \) is a regular point – the gradients of the active constraints at \( x^* \), \( \{ \nabla g_i(x^*) : g_i(x^*) = 0 \} \), are linearly independent,
3. \( \nabla_x \mathcal{L}(x^*, \lambda^*) = 0 \), where \( \nabla_x \) denotes the gradient with respect to \( x \) and \( \mathcal{L}(x, \lambda) = f(x) - x^T g(x) \),
4. \( \lambda^* \geq 0 \),
5. $\lambda^T g(x^*) = 0$ (complementary slackness condition), and
6. $Z(x^*)^T \nabla^2_{xx} L(x^*, \lambda^*) Z(x^*)$ is positive definite,

where $Z$ is a basis for the null space of the Jacobian matrix $\nabla g(x)^T$ of the active constraints with positive Lagrange multipliers at $x^*$.

Note that for our maximization problem, Condition 6 will correspond to the quantity $Z(x^*)^T \nabla^2_{xx} L(x^*, \lambda^*) Z(x^*)$ being negative definite. The remaining conditions are equivalent between the problems of minimization and maximization.

A.2 A Solution

First, let us begin with the first order condition $\nabla P L(P^*, \lambda^*) = 0$ in order to find an optimal result. Due to the symmetry of the problem, all of the entries of $\nabla P L(P, \lambda)$ will have the same form. Consequently, we can simply take the partial derivative of (A.4) with respect to $P_i$:

$$\frac{\partial}{\partial P_i} L(P, \lambda) = \frac{1}{2M} \left[ \frac{1}{P_i + N_i + N_s} + (M - 1) \frac{1}{P_i + N_i} \right] \lambda_0 - \lambda_i$$

where we will define $\mu_i = \lambda_0 - \lambda_i$ to simplify notation. The result in (A.5) can be obtained by expanding the sums in (A.4) and taking the partial derivative with respect to $P_1$ without loss of generality.

Setting (A.5) equal to zero gives us:

$$\frac{\partial}{\partial P_i} L(P, \lambda) = \frac{1}{2M} \left[ \frac{1}{P_i + N_i + N_s} + (M - 1) \frac{1}{P_i + N_i} \right] + \mu_i = 0$$

$$\Rightarrow P_i + N_i + (M - 1)(P_i + N_i + N_s) + 2M \mu_i (P_i + N_i) (P_i + N_i + N_s) = 0$$

$$\Rightarrow M(P_i + N_i) + (M - 1) N_s + 2M \mu_i (N_i^2 + 2P_i N_i + P_i^2 + N_s (P_i + N_i)) = 0$$

$$\Rightarrow 2M \mu_i P_i^2 + (M + 2M \mu_i (2N_i + N_s)) P_i + \ldots$$

$$+ (MN_i + (M - 1) N_s + 2M \mu_i (N_i^2 + N_i N_s)) = 0$$
\[ P_i^2 + \left( 2N_i + N_s + \frac{1}{2\mu_i} \right) P_i + \left( N_i^2 + N_s N_i + \frac{1}{2\mu_i} N_i + \frac{1}{\mu_i} \frac{M-1}{2M} N_s \right) = 0 \]  
(A.6e)

\[ P_i = -\frac{1}{2} \left( 2N_i + N_s + \frac{1}{2\mu_i} \right) \ldots \]

\[ \pm \frac{1}{2} \sqrt{\left( 2N_i + N_s + \frac{1}{2\mu_i} \right)^2 - 4 \left( N_i^2 + N_s N_i + \frac{1}{2\mu_i} N_i + \frac{1}{\mu_i} \frac{M-1}{2M} N_s \right)} \]

(A.6f)

\[ P_i = -N_i - \frac{1}{2} \left( N_s + \frac{1}{2\mu_i} \right) \ldots \]

\[ \pm \frac{1}{2} \sqrt{\left( 2N_i + N_s + \frac{1}{2\mu_i} \right)^2 - 4 \left( N_i^2 + N_s N_i + \frac{1}{2\mu_i} N_i + \frac{1}{\mu_i} \frac{M-1}{2M} N_s \right)} \]

(A.6g)

\[ P_i = -N_i + \frac{-N_s - \frac{1}{2\mu_i} \pm \sqrt{N_s^2 - \frac{M-2}{M} N_s + \frac{1}{4\mu_i}}}{\nu} \]

(A.6h)

\[ P_i = \nu - N_i. \]  
(A.6i)

Let us briefly consider the range of \( \nu \) as a function of \( \mu_i \). First, we note that since \( P_i = \nu - N_i \geq 0 \) and \( N_i \geq 0 \), we can say \( \nu \geq \nu - N_i \geq 0 \). Consequently, we need \( \nu \) to have a minimum value of 0 and an unbounded maximum (\( \infty \)). For simplicity, we will only consider the case in which the radical term in (A.6h) is added and ignore the case in which the radical term is subtracted.

Since \( \lambda_0 \) and \( \lambda_i \geq 0 \) in accord with the Kuhn-Tucker condition \( \lambda_i \geq 0 \), \( \mu_i = \lambda_0 - \lambda_i \) is unrestricted and can take on any value. As \( \mu_i \to 0^- \), we see \( \nu \to +\infty \). For the other extreme, as \( \mu_i \to -\infty \),

\[ \nu \to \frac{1}{2} \left( -N_s + \sqrt{N_s^2} \right) \]

\[ = \frac{1}{2} (-N_s + N_s) \]

\[ = 0. \]

(A.7)
Hence, by varying $\mu_i \in (-\infty, 0)$, we have $\nu \in (0, \infty)$. See Figure A.1 for more details.

Figure A.1. The variable $\nu$ as a function of $\mu_i$. The solid and dashed lines correspond to the cases in which the square root term is added and subtracted, respectively.

Continuing from (A.6i), we again note that the constraint in (A.3b) that all $P_i \geq 0$ will require us to modify our answer to

$$P_i = (\nu - N_i)^+,$$  \hspace{1cm} (A.8)
where we define
\[
y^+ = \begin{cases} 
y, & y > 0 \\
0, & y \leq 0.
\end{cases}
\] (A.9)

The result in (A.8) is traditionally called “water-filling” since the power is distributed among noisy channels as water fills empty space in a container. The value of \( \nu \) is chosen to satisfy \( \sum_i P_i \leq P \).

A.3 Satisfying Conditions 1 and 2

Now let us return to the Kuhn-Tucker conditions listed in Theorem 1. We are selecting \( \nu \) according to (A.8) such that the constraints given by Condition 1 are satisfied. To show Condition 2 is satisfied, let us consider the case with the highest possible number of active constraints. In this case, all but one channel will have \( P_i = 0 \), while the remaining channel will have \( P_i = P \). Without loss of generality, let us say channel \( i = 1 \) has power \( P \) while the remaining \( M - 1 \) channels have zero power. Our Jacobian constraint matrix is given by
\[
\nabla g(P)^T = \begin{bmatrix}
1 & 1 & 1 & 1 & \cdots & 1 \\
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & 1
\end{bmatrix}. \tag{A.10}
\]

Since the largest possible \( \nabla g(P)^T \) has \( M \) linearly independent columns, Condition 2 is satisfied. Any smaller Jacobian constraint matrix will simply have rows missing, but the linear independence of the remaining rows will remain.
A.4 Satisfying Condition 3

Next, we turn our attention to Condition 3. We will show (A.8) satisfies the condition \( \nabla_P \mathcal{L}(P_s, \lambda_s) = 0 \) by considering separately the case in which \( P_i = \nu - N_i > 0 \) and the case in which \( P_i = 0 \).

First, we substitute \( P_i = \nu - N_i > 0 \) into (A.5):

\[
\begin{align*}
\frac{1}{2M} \left[ \frac{1}{(\nu - N_i) + N_i + N_s} + (M - 1) \frac{1}{(\nu - N_i) + N_i} \right] + \mu_i &= 0 \quad \text{(A.11a)} \\
\Rightarrow \frac{1}{\nu + N_s} + \frac{M - 1}{\nu} + 2M\mu_i &= 0 \quad \text{(A.11b)} \\
\Rightarrow &\nu + (M - 1)(\nu + N_s) + 2M\mu_i(\nu^2 + N_s\nu) = 0 \quad \text{(A.11c)} \\
\Rightarrow &2M\mu_i\nu^2 + (2M\mu_iN_s + M)\nu + (M - 1)N_s = 0. \quad \text{(A.11d)}
\end{align*}
\]

Next, from (A.6h) and the analysis shown in Figure A.1, we simply \( \nu \) to

\[
\nu = \frac{1}{2} \left( -N_s - \frac{1}{2\mu_i} + \sqrt{N_s^2 - \frac{1}{\mu_i} M - 2 M N_s + \frac{1}{4\mu_i^2}} \right)
= \frac{1}{2} \left( -N_s - \frac{1}{2\mu_i} + \beta \right), \quad \text{(A.12)}
\]

where \( \beta \) is defined as the square root term. We substitute (A.12) into (A.11d) to get

\[
\begin{align*}
2M\mu_i &\left( -N_s - \frac{1}{2\mu_i} + \beta \right)^2 + (2M\mu_iN_s + M)\frac{1}{2} \left( -N_s - \frac{1}{2\mu_i} + \beta \right) \ldots \\
+ (M - 1)N_s &= 0 \quad \text{(A.13a)} \\
\Rightarrow &\frac{1}{2} M\mu_i \left( N_s^2 + \frac{N_s}{\mu_i} - 2N_s\beta + \frac{1}{4\mu_i^2} - \frac{\beta}{\mu_i} + \beta^2 \right) \ldots \\
- \left( M\mu_iN_s^2 + MN_s - M\mu_iN_s\beta + \frac{M}{4\mu_i^2} - \frac{M\beta}{2} \right) + (M - 1)N_s &= 0. \quad \text{(A.13b)}
\end{align*}
\]
Here we note

\[ \beta^2 = N_s^2 - \frac{1}{\mu_i} \frac{M - 2}{M} N_s + \frac{1}{4\mu_i^2}. \]  

(A.14)

Substituting (A.14) into (A.13b) gives us

\[
\frac{1}{2} M \mu_i \left( 2N_s^2 + \frac{2}{M \mu_i} N_s - 2N_s \beta + \frac{1}{2\mu_i^2} - \frac{\beta}{\mu_i} \right) \ldots 
- \left( M \mu_i N_s^2 + MN_s - M \mu_i N_s \beta + \frac{M}{4\mu_i} - \frac{M \beta}{2} \right) + (M-1) N_s = 0 \quad \text{(A.15a)}
\]

\[ \Rightarrow N_s - MN_s + (M-1)N_s = 0 \quad \text{(A.15b)} \]

\[ \Rightarrow MN_s - MN_s \not\equiv 0. \quad \text{(A.15c)} \]

Therefore, the solution \( P_i = \nu - N_i > 0 \) satisfies Condition 3.

Now let us consider the case \( P_i = 0 \). This is equivalent to saying \( \nu = N_i \). Since \( \nu \) is a function of \( \mu_i \), our approach is to show that the \( \tilde{\mu}_i \) that results in \( \nu(\tilde{\mu}_i) = N_i \) also satisfies Condition 3 when \( P_i = 0 \). First, let us solve for \( \mu_i \) from (A.5), setting \( P_i = 0 \):

\[
\frac{1}{2} M \mu_i \left( N_i + N_s + \frac{M - 1}{N_i} \right) + \mu_i = 0 \quad \text{(A.16a)}
\]

\[ \Rightarrow \mu_i = \frac{1}{2M} \left( \frac{N_i + (M - 1)(N_i + N_s)}{N_i(N_s + N_i)} \right) \quad \text{(A.16b)} \]

\[ \Rightarrow \mu_i = -\frac{MN_i + (M - 1)N_s}{2MN_i(N_s + N_i)}. \quad \text{(A.16c)} \]

Next, let us find the \( \tilde{\mu}_i \) such that \( \nu(\tilde{\mu}_i) = N_i \). From (A.6g), we have

\[
\nu(\tilde{\mu}_i) = N_i = -\frac{1}{2} \left( N_s + \frac{1}{2\tilde{\mu}_i} \right) \ldots 
+ \frac{1}{2} \sqrt{\left( 2N_i + N_s + \frac{1}{2\tilde{\mu}_i} \right)^2 - 4 \left( N_i^2 + N_s N_i + \frac{1}{2\tilde{\mu}_i} N_i + \frac{1}{\mu_i} \frac{M - 1}{2M} N_s \right)}. \quad \text{(A.17a)}
\]
\[ 2N_i + N_s + \frac{1}{2\mu_i} = \ldots \]
\[ \sqrt{\left(2N_i + N_s + \frac{1}{2\mu_i}\right)^2 - 4 \left(N_i^2 + N_s N_i + \frac{1}{2\mu_i} N_i + \frac{1}{\mu_i} \frac{M-1}{2M} N_s\right)} \]  
(A.17b)

\[ \Rightarrow \left(2N_i + N_s + \frac{1}{2\mu_i}\right)^2 = \ldots \]
\[ \left(2N_i + N_s + \frac{1}{2\mu_i}\right)^2 - 4 \left(N_i^2 + N_s N_i + \frac{1}{2\mu_i} N_i + \frac{1}{\mu_i} \frac{M-1}{2M} N_s\right) \]  
(A.17c)

\[ \Rightarrow 0 = N_i^2 + N_s N_i + \frac{1}{\mu_i} \left(\frac{1}{2} N_i + \frac{M-1}{2M} N_s\right) \]  
(A.17d)

\[ \Rightarrow \frac{1}{\mu_i} = -\frac{N_i^2 + N_s N_i}{\frac{1}{2} N_i + \frac{M-1}{2M} N_s} = -\frac{2MN_i (N_i + N_s)}{MN_i + (M-1)N_s} \]  
(A.17e)

\[ \Rightarrow \mu_i = -\frac{MN_i + (M-1)N_s}{2MN_i (N_i + N_s)} \]  
(A.17f)

Thus, from the equality between (A.16c) and (A.17f), we conclude that \( P_i = 0 \) can also satisfy Condition 3. Hence, the power distribution given by \( P_i = (\nu - N_i)^+ \) satisfies Condition 3.

A.5 Satisfying Conditions 4 and 5

We have assumed throughout our analysis that \( \lambda_0 \geq 0 \) and \( \lambda_i \geq 0 \), which satisfies Condition 4.

To satisfy complementary slackness (Condition 5), we note that if a particular constraint \( g_i(P) \) is active, then \( g_i(P) = 0 \) and the condition is satisfied. Otherwise, when the constraint is not active, \( \lambda_i = 0 \). Therefore, the value of \( \mu_i = \lambda_0 - \lambda_i \) may be restricted in some cases.

In the case \( P_i = \nu - N_i > 0 \), the nonnegative constraint on \( P_i \) is not active (\( i.e. \ g_i(P) \neq 0 \)). Consequently, \( \lambda_i = 0 \) in this case, which means \( \mu_i = \lambda_0 \geq 0 \). However,
from the analysis concluded by (A.15c), we see that the constraint on $\mu_i$ does not affect optimality.

In the case $P_i = 0$, the nonnegative constraint is active and $\mu_i = \lambda_0 - \lambda_i$ is unrestricted. Hence, (A.17f) holds for any parametric values.

Thus, Condition 5 is satisfied.

A.6 Satisfying Condition 6

To satisfy Condition 6 in our problem, we need $Z(P_\ast)^T \nabla^2_{PP} \mathcal{L}(P_\ast, \lambda_\ast) Z(P_\ast)$ to be negative definite. First, let us consider the case with the highest possible number of active constraints as described by the Jacobian constraint matrix given in (A.10).

Since $\nabla g(P)^T$ has full rank, the null space of $\nabla g(P)^T$ is empty, which means $Z$ is empty. In this case, Condition 6 is trivially satisfied [28, Ex. 14.20].

Next, let us consider the case with fewest possible active constraints. In such a case, all $P_i > 0$ and the only active constraint is $\sum_{i=1}^{M} P_i = P$. Hence, our Jacobian constraint matrix is given by $\nabla g(P)^T = [1 \ldots 1]$. Since $\nabla g(P)^T$ has a rank of 1, a basis for the null space will have a rank of $\left(M - \text{rank}(\nabla g(P)^T)\right) = M - 1$. We desire $\nabla g(P)^T Z = 0$, so for simplicity, we need $M - 1$ linearly independent columns

\[1\]This the only optimal case of one active constraint. To prove this, we simply use contradiction. We assume that the optimal solution involves a set of $P_i$ such that $\sum_i P_i < P$. Since the constraint is not active, we can increase the power on a subchannel that is in use. The resulting increase in SNR ($P_i/N_i$) increases the mutual information, which contradicts our assumption that the optimal solution follows $\sum_i P_i < P$. Therefore, the optimal solution has the active constraint $\sum_i P_i = P$.  

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containing a single 1 and −1. One possible basis is the \( M \times (M - 1) \) matrix
\[
\mathbf{Z} = \begin{bmatrix}
1 \\
-1 & 1 \\
-1 & 1 \\
\vdots & \vdots \\
\vdots & \vdots \\
-1 & 1 \\
-1 \\
\end{bmatrix}. \tag{A.18}
\]
This choice satisfies \( \nabla g(P)^T \mathbf{Z} = 0 \). Next, we turn our attention to \( \nabla^2_{PP} \mathcal{L}(P, \lambda) \).

From (A.5), we see that \( \frac{\partial^2}{\partial P_i \partial P_k} \mathcal{L}(P, \lambda) = 0 \) for all \( k \neq i \). Consequently, \( \nabla^2_{PP} \mathcal{L}(P, \lambda) \) is a diagonal matrix with the \( i \)th diagonal entry equal to
\[
\frac{\partial^2}{\partial P_i^2} \mathcal{L}(P, \lambda) = -\frac{1}{2M} \left[ \frac{1}{(P_i + N_i + N_s)^2} + \frac{M - 1}{(P_i + N_i)^2} \right]. \tag{A.19}
\]
For the two possible optimal solutions \((P_i = 0 \text{ or } P_i = \nu - N_i)\), we have
\[
\frac{\partial^2}{\partial P_i^2} \mathcal{L}(P, \lambda) \bigg|_{P_i = \nu - N_i} = -\frac{1}{2M} \left[ \frac{1}{(\nu + N_s)^2} + \frac{M - 1}{\nu^2} \right] \tag{A.20a}
\]
\[
\Delta \equiv \alpha < 0 \tag{A.20b}
\]
and
\[
\frac{\partial^2}{\partial P_i^2} \mathcal{L}(P, \lambda) \bigg|_{P_i = 0} = -\frac{1}{2M} \left[ \frac{1}{(N_i + N_s)^2} + \frac{M - 1}{N_i^2} \right] \tag{A.21a}
\]
\[
\Delta \equiv \gamma_i < 0, \tag{A.21b}
\]
where we define \( \alpha \) and \( \gamma_i \) above to simplify notation. Note that \( \alpha \) and \( \gamma_i \) are negative values.
In the case with the fewest active constraints, \( P_{si} = \nu - N_i, \forall i \). Consequently, \( \nabla_{P_{si}}^2 \mathcal{L}(P_{si}, \lambda_s) = \alpha I \), where \( I \) is an \( M \times M \) identify matrix. Consequently, Condition 6 results in

\[
Z^T \nabla_{P_{si}}^2 \mathcal{L}(P_{si}, \lambda_s) Z = Z^T \alpha I Z
\]

\[
= \alpha Z^T Z \tag{A.22a}
\]

\[
= \alpha \begin{bmatrix}
1 & -1 \\
1 & -1 \\
\vdots & \vdots \\
1 & -1
\end{bmatrix}_{(M-1) \times M} \begin{bmatrix}
1 \\
-1 \\
\vdots \\
-1
\end{bmatrix}_{M \times (M-1)} \tag{A.22b}
\]

\[
= \alpha \begin{bmatrix}
2 & -1 & & & & & \\
-1 & 2 & -1 & & & & \\
& -1 & \ddots & \ddots & & & \\
& & \ddots & \ddots & \ddots & & \\
& & & \ddots & \ddots & \ddots & \\
& & & & \ddots & \ddots & \ddots \\
& & & & & -1 & 2
\end{bmatrix}_{(M-1) \times (M-1)} \tag{A.22c}
\]

Now we need to determine if the result in \( \text{(A.22d)} \) is negative definite. A negative definite matrix \( A \) satisfies \( y^T A y < 0, \forall y \neq 0 \). Hence, we have

\[
\alpha \begin{bmatrix}y_1, \ldots, y_{M-1}\end{bmatrix} \begin{bmatrix}
2 & -1 & & & & & \\
-1 & 2 & -1 & & & & \\
& -1 & \ddots & \ddots & & & \\
& & \ddots & \ddots & \ddots & & \\
& & & \ddots & \ddots & \ddots & \\
& & & & \ddots & \ddots & \ddots \\
& & & & & -1 & 2
\end{bmatrix} \begin{bmatrix}y_1 \\
\vdots \\
y_{M-1}\end{bmatrix} < 0 \tag{A.23a}
\]
\[
\begin{pmatrix}
2y_1 - y_2 \\
-y_1 + 2y_2 - y_3 \\
\vdots \\
-y_{M-3} + 2y_{M-2} - y_{M-1} \\
-y_{M-2} + 2y_{M-1}
\end{pmatrix}
\]

\(= \alpha [y_1, \ldots, y_{M-1}] \) \hfill (A.23b)

\[
= \alpha \left( 2y_1^2 - y_1y_2 - y_1y_2 + 2y_2^2 - y_2y_3 \ldots \\
- y_{M-3}y_{M-2} + 2y_{M-2}^2 - y_{M-2}y_{M-1} - y_{M-2}y_{M-1} + 2y_{M-1}^2 \right) \) \hfill (A.23c)

\[
= \alpha \left( y_1^2 + (y_1 - y_2)^2 + \ldots + (y_{M-2} - y_{M-1})^2 + y_{M-1}^2 \right) \) \hfill (A.23d)

\[
< 0 \) \hfill (A.23e)
\]

because \((y_1^2 + (y_1 - y_2)^2 + \ldots + (y_{M-2} - y_{M-1})^2 + y_{M-1}^2) > 0\) and \(\alpha < 0\) from (A.20b).

Therefore, the matrix in (A.22d) is negative definite, and Condition [6] is satisfied.

For the cases between these two extremes (i.e. cases with more than one active constraint but fewer than \(M\)), we offer the following inductive argument. First, each additional constraint will remove a column from (A.18). In other words, for \(\nabla g(P)^T\) of size \(r \times M\), \(Z\) will have dimension \(M \times (M - r)\).

In addition to fewer columns, for each active constraint \(P_i = 0\), we will require \(Z\) to have zeros in the \(i\)th row. This will ultimately result in the \(i\)th diagonal element of \(\nabla^2_{PP}L(P^*, \lambda^*)\) being multiplied by zero and eliminating the element from consideration. As before, each column will have a single 1 and \(-1\).

We concisely summarize these rules with the following algorithm for the construction of \(Z\):

**Algorithm 1.** Constructing a basis for the null space of \(\nabla g(P)^T\):

1. For \(r\) active constraints, construct \(Z\) to be an \(M \times (M - r)\) matrix.
2. For all active constraints \(P_i = 0\), fill the \(i\)th row of \(Z\) with zeros.
3. In the first column of $Z$, enter a 1 in the first nonzero row from the top. Enter a $-1$ in the second nonzero row from the top.

4. In the next column, enter a 1 in the row for which a $-1$ was entered in the previous column. Enter a $-1$ in the next available nonzero row down the column.

5. Repeat Step 4 until all columns of $Z$ have a single 1 and $-1$.

With a $Z$ constructed according to Algorithm 1, we can show $Z^T \nabla_{PP}^2 \mathcal{L}(P^*, \lambda^*) Z$ is negative definite by following the same analysis given above for the case of the fewest possible constraints. We present a brief example for illustration.

**Example A.6.1.** For this example, we have $M = 4$. Let us consider three scenarios given in Table A.1. In scenario A, only the sum-power constraint ($\sum_i P_i = P$) is active. Scenario B adds $P_2 = 0$, and finally, Scenario C adds the constraint $P_4 = 0$. The Jacobian constraint matrices and the corresponding $Z$ matrices based on Algorithm 1 are shown in Table A.1.

We have already shown (for general $M$) that Condition 6 for Scenario A results in a negative definite matrix (see analysis beginning with (A.22)). For Scenario B,

\[
Z^T \nabla_{PP}^2 \mathcal{L}(P^*, \lambda^*) Z = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \gamma_2 \\ \alpha \\ \alpha \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}
\]

(A.24a)

\[
= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & 0 \\ -\alpha & \alpha \\ 0 & -\alpha \end{bmatrix}
\]

(A.24b)

\[
= \begin{bmatrix} 2\alpha & -\alpha \\ -\alpha & 2\alpha \end{bmatrix} = \alpha \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.
\]

(A.24c)
### Table A.1

The Jacobian constraint matrices and $Z$ matrices for three scenarios of active constraints

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\nabla g(P)^T = \begin{bmatrix} 1 &amp; 1 &amp; 1 &amp; 1 \end{bmatrix}$</td>
<td>$\nabla g(P)^T = \begin{bmatrix} 1 &amp; 1 &amp; 1 &amp; 1 \ 0 &amp; 1 &amp; 0 &amp; 0 \end{bmatrix}$</td>
<td>$\nabla g(P)^T = \begin{bmatrix} 1 &amp; 1 &amp; 1 \ 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 \end{bmatrix}$</td>
</tr>
<tr>
<td></td>
<td>$r = 1$</td>
<td>$r = 2$</td>
<td>$r = 3$</td>
</tr>
<tr>
<td></td>
<td>$Z = \begin{bmatrix} 1 &amp; 0 &amp; 0 \ -1 &amp; 1 &amp; 0 \ 0 &amp; -1 &amp; 1 \ 0 &amp; 0 &amp; -1 \end{bmatrix}$</td>
<td>$Z = \begin{bmatrix} 1 &amp; 0 \ 0 &amp; 0 \ -1 &amp; 1 \ 0 &amp; -1 \end{bmatrix}$</td>
<td>$Z = \begin{bmatrix} 1 \ 0 \ -1 \ 0 \end{bmatrix}$</td>
</tr>
<tr>
<td></td>
<td>$(4 \times 3)$</td>
<td>$(4 \times 2)$</td>
<td>$(4 \times 1)$</td>
</tr>
</tbody>
</table>

which we know is negative definite from the analysis given by (A.23). In Scenario C,

$$Z^T \nabla^2_{pp} \mathcal{L}(P_*, \Lambda_*) Z = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \gamma_2 \\ \alpha \\ \gamma_4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$ (A.25a)
\[
\begin{bmatrix}
1 & 0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
\alpha \\
0 \\
-\alpha \\
0
\end{bmatrix}
= 2\alpha < 0,
\]

so Condition 6 is satisfied. ■

A.7 Global Maximum

Since the water-filling solution \( P_i = (\nu - N_i)^+ \) satisfies the Kuhn-Tucker conditions, the solution is proven to be a local maximum. If we can show that the objective function in (A.2) is a concave function over a convex set, then the solution is also a global solution [28, Th. 2.1].

We know that linear constraints result in a convex set, and the intersection of a finite number of convex sets is also a convex set [28, Exercise 3.1]. Consequently, our constraints in (A.3) result in a feasibility region that is a convex set.

We can show that the objective function \( f(P) \) is concave by showing that \( \nabla^2 f(P) \) is negative semidefinite. Since the Lagrangian multipliers are only first order terms in the Lagrangian function \( \mathcal{L}(P, \lambda) \) in (A.4), \( \nabla^2 f(P) = \nabla^2_{PP}\mathcal{L}(P, \lambda) \). Consequently, \( \nabla^2 f(P) \) is the diagonal matrix

\[
\nabla^2 f(P) = \\
\begin{bmatrix}
\xi_1 \\
\xi_2 \\
\vdots \\
\xi_M
\end{bmatrix},
\] (A.26)
where

\[
\xi_i = -\frac{1}{2M} \left[ \frac{1}{(P_i + N_i + N_s)^2} + \frac{M - 1}{(P_i + N_i)^2} \right] < 0.
\]  

(A.27)

Since the eigenvalues of a diagonal matrix are the diagonal elements themselves, we see that all of the eigenvalues of \(\nabla^2 f(P)\) are negative. Therefore, \(\nabla^2 f(P)\) is negative definite by definition, and \(f(P)\) is a strictly concave function.\(^2\)

Since \(f(P)\) is a strictly concave function over a convex set, any local maximum is also the unique global maximum [28, Th. 2.1]. Hence, our solution \(P_i = (\nu - N_i)^+\) is the unique global maximizer of our problem of interest. □

\(^2\)For a function \(f\) to be concave, \(\nabla^2 f\) need only be negative semidefinite. If \(\nabla^2 f\) is negative definite, we say \(f\) is strictly concave.
**APPENDIX B**

**ZEROTH AND SECOND ORDER APPROXIMATION OF $h(X)$**

**B.1 Zeroth Order Approximation $h_0(X)$**

To calculate the zeroth order approximation, we use (3.10) with $R = 0$ to obtain

$$h_0(X) \approx -\sum_{i=1}^{L} \int_{\mathbb{R}^N} \omega_i \cdot \mathcal{N}(\bar{x}; \mu_i, C_i) \cdot \log(f(\mu_i)) \, d\bar{x}$$

$$= -\sum_{i=1}^{L} \omega_i \cdot \log(f(\mu_i)) \int_{\mathbb{R}^N} \mathcal{N}(\bar{x}; \mu_i, C_i) \, d\bar{x}$$

$$= -\sum_{i=1}^{L} \omega_i \cdot \log(f(\mu_i)).$$

(B.1)

Since the first central moment of a Gaussian is zero, the first order approximation is equivalent to the zeroth order approximation (*i.e.* $h_1(X) = h_0(X)$).

**B.2 Second Order Approximation $h_2(X)$**

In deriving the second order approximation, it is useful to know the first and second partial derivatives of $\log(f(\bar{x}))$:

$$\frac{\partial}{\partial x_j} \log(f(\bar{x})) = \sum_{i=1}^{L} \frac{\partial f(\bar{x})}{\partial x_j} \omega_i \cdot \mathcal{N}(\bar{x}; \mu_i, C_i) \cdot \left( -C_i^{-1}(j)(\bar{x} - \mu_i) \right)$$

$$\sum_{i=1}^{L} \omega_i \cdot \mathcal{N}(\bar{x}; \mu_i, C_i)$$

(B.2)
\[
\frac{\partial^2}{\partial x_j \partial x_k} \log(f(\tilde{x})) = \frac{f(\tilde{x}) \frac{\partial^2}{\partial x_j \partial x_k} f(\tilde{x}) - \frac{\partial}{\partial x_j} f(\tilde{x}) \frac{\partial}{\partial x_k} f(\tilde{x})}{f(\tilde{x})^2},
\]

(B.3)

where \( \frac{\partial}{\partial x_j} f(\tilde{x}) \) is the numerator of the fraction as shown in (B.2) with index \( j \), and the notation \( K(j) \) refers to the \( j \)th row of the matrix \( K \). For convenience, we write out

\[
\frac{\partial}{\partial x_j} f(\tilde{x}) = \sum_{i=1}^{L} \omega_i \cdot N(\tilde{x}, \mu_i, C_i) \cdot \left(-C_i^{-1}(j)(\tilde{x} - \mu_i)\right)
\]

(B.4)

\[
\frac{\partial^2}{\partial x_j \partial x_k} f(\tilde{x}) = \sum_{i=1}^{L} \omega_i \cdot N(\tilde{x}, \mu_i, C_i) \cdot \left(C_i^{-1}(j)(\tilde{x} - \mu_i)C_i^{-1}(k)(\tilde{x} - \mu_i) - C_i^{-1}_{j,k}\right).
\]

(B.5)

To simplify the complicated notation, we define \( \eta(i, j, k) \) to be (B.3) evaluated at \( \tilde{x} = \mu_i \). Mathematically,

\[
\eta(i, j, k) = \left. \frac{f(\tilde{x}) \frac{\partial^2}{\partial x_j \partial x_k} f(\tilde{x}) - \frac{\partial}{\partial x_j} f(\tilde{x}) \frac{\partial}{\partial x_k} f(\tilde{x})}{f(\tilde{x})^2} \right|_{\tilde{x} = \mu_i},
\]

(B.6)

where (B.4), (B.5), and (3.8) complete the equation.

Let us now derive the second order approximation by setting \( R = 2 \) in (3.10):

\[
\begin{aligned}
&h_2(X) \approx - \sum_{i=1}^{L} \int_{\mathbb{R}^N} \omega_i \cdot N(\tilde{x}; \mu_i, C_i) \cdot \left[ \log(f(\tilde{x})) \ldots \\
&\quad + \left((x_1 - \mu_{i,1}) \frac{\partial}{\partial x_1} + \ldots + (x_n - \mu_{i,n}) \frac{\partial}{\partial x_n}\right) \log(f(\tilde{x})) \ldots \\
&\quad + \frac{1}{2} \left((x_1 - \mu_{i,1}) \frac{\partial}{\partial x_1} + \ldots + (x_n - \mu_{i,n}) \frac{\partial}{\partial x_n}\right)^2 \log(f(\tilde{x})) \right] \right|_{\tilde{x} = \mu_i} \mathbb{dx} \\
&= - \sum_{i=1}^{L} \omega_i \left[ \log(f(\mu_i)) \int_{\mathbb{R}^N} N(\tilde{x}; \mu_i, C_i) \mathbb{dx} \ldots \right]
\end{aligned}
\]

(B.7a)
\[ + \int_{\mathbb{R}^N} \mathcal{N}(\bar{x}; \mu_i, C_i) \left( \left( (x_1 - \mu_{i,1}) \frac{\partial}{\partial x_1} + \ldots + (x_n - \mu_{i,n}) \frac{\partial}{\partial x_n} \right) \log(f(\bar{x})) \right) \Bigg|_{\bar{x} = \mu_i} \, d\bar{x} \ldots \]

\[ + \int_{\mathbb{R}^N} \mathcal{N}(\bar{x}; \mu_i, C_i) \frac{1}{2} \left( (x_1 - \mu_{i,1}) \frac{\partial}{\partial x_1} + \ldots + (x_n - \mu_{i,n}) \frac{\partial}{\partial x_n} \right)^2 \log(f(\bar{x})) \Bigg|_{\bar{x} = \mu_i} \, d\bar{x} \]

\[ = h_0(\mathcal{X}) \]

\[ = - \sum_{i=1}^{L} \omega_i \log(f(\mu_i)) \ldots \]

\[ - \sum_{i=1}^{L} \frac{\omega_i}{2} \int_{\mathbb{R}^N} \mathcal{N}(\bar{x}; \mu_i, C_i) \cdot \sum_{j=1}^{n} \sum_{k=1}^{n} (x_j - \mu_{i,j})(x_k - \mu_{i,k}) \frac{\partial^2}{\partial \bar{x}_j \partial \bar{x}_k} \log(f(\bar{x})) \Bigg|_{\bar{x} = \mu_i} \, d\bar{x} \]

\[ = h_0(\mathcal{X}) - \sum_{i=1}^{L} \frac{\omega_i}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \eta(i,j,k) \frac{\partial^2}{\partial \bar{x}_j \partial \bar{x}_k} \log(f(\bar{x})) \Bigg|_{\bar{x} = \mu_i} \ldots \]

\[ \cdot \int_{\mathbb{R}^N} \mathcal{N}(\bar{x}; \mu_i, C_i) (x_j - \mu_{i,j})(x_k - \mu_{i,k}) d\bar{x} \]

\[ = h_0(\mathcal{X}) - \sum_{i=1}^{L} \frac{\omega_i}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \eta(i,j,k) C_{i,jk}. \]

We also note that the third order approximation is equivalent to \( h_2(\mathcal{X}) \).
BIBLIOGRAPHY


