1. Determine the values of total angular momentum of a $2p$ state (orbital angular momentum $l = 1$) of hydrogen and write out the angular wave functions for each of the 6 possible substates.

   There are 2 possible values of j: $j = 1/2,\ 3/2$. The corresponding angular wave functions are

$$
\begin{aligned}
\Omega_{3/2,1,3/2} &= Y_{1,1}(\theta,\phi)\chi_{1/2} \\
\Omega_{3/2,1,1/2} &= \sqrt{2/3}\ Y_{1,0}(\theta,\phi)\chi_{1/2} + \sqrt{1/3}\ Y_{1,1}(\theta,\phi)\chi_{-1/2} \\
\Omega_{3/2,1,-1/2} &= \sqrt{1/3}\ Y_{1,-1}(\theta,\phi)\chi_{1/2} + \sqrt{2/3}\ Y_{1,0}(\theta,\phi)\chi_{-1/2} \\
\Omega_{3/2,1,3/2} &= Y_{1,-1}(\theta,\phi)\chi_{-1/2}
\end{aligned}
$$

$$
\begin{aligned}
\Omega_{1/2,1,1/2} &= -\sqrt{1/3}\ Y_{1,0}(\theta,\phi)\chi_{1/2} + \sqrt{2/3}\ Y_{1,1}(\theta,\phi)\chi_{-1/2} \\
\Omega_{3/2,1,-1/2} &= -\sqrt{2/3}\ Y_{1,-1}(\theta,\phi)\chi_{1/2} + \sqrt{1/3}\ Y_{1,0}(\theta,\phi)\chi_{-1/2}
\end{aligned}
$$

2. Determine the number of ground-state hyperfine levels of the hydrogen-like ion $^{209}\mathrm{Bi}^{+82}$. Find the eigenvalue f of total angular momentum $\boldsymbol{F} = \boldsymbol{I} + \boldsymbol{J}$ for each level and give its degeneracy.

   Given that $j = 1/2$ and $i = 9/2$ there are a total of $(2j + 1)(2i + 1) = 20$ sublevels. These combine to give **two** possible eigenstates of total angular momentum $f = 4$ and 5. The level $f = 4$ has 9 degenerate sublevels and the level $f = 5$ has 11 possible sublevels.

3. Prove [Varshalovich, Sec. 5.9.1, Eq.(5)]

$$
\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\ Y_{l_1,m_1}(\theta,\phi)\, Y_{l_2m_2}(\theta,\phi)\, Y_{l_3m_3}(\theta,\phi)
= \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}
$$

   Rewrite Eq.(1.97) in the notes as

$$
Y_{l_2m_2}(\Omega)Y_{l_3m_3}(\Omega) = \sum_{l_4} \sqrt{\frac{2l_2+1}{4\pi}} (-1)^{l_4-m_4} \begin{pmatrix} l_4 & l_2 & l_3 \\ -m_4 & m_2 & m_3 \end{pmatrix} \langle l_4 \| C^{l_2} \| l_2 \rangle Y_{l_4m_4}(\Omega)
$$

   Multiply both sides of this equation by $Y_{l_1m_1}(\Omega)$ and integrate over $d\Omega$. The l.h.s. of the resulting equation is the desired answer. The angular integral on the r.h.s. is

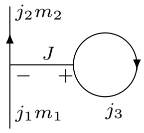
$$
\int d\Omega\ Y_{l_1m_1}(\Omega) Y_{l_4m_4}(\Omega) = (-1)^{m_1} \delta_{l_4,l_1} \delta_{m_4,-m_1}
$$

   The sum on the r.l.s. reduces to a single term

$$
\sqrt{\frac{2l_2+1}{4\pi}} (-1)^{l_1} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \langle l_1 \| C^{l_2} \| l_2 \rangle
$$

   Substituting the expression for the reduced martix element from Eq.(1.99) leads to the desired result.

4. Derive the graphical identity



$$= \delta_{j_1 j_2}\, \delta_{m_1 m_2}\, \delta_{J0} \sqrt{\frac{2j_3+1}{2j_1+1}}$$

The mathematical expression for this graph is

$$\sum_{M,m_3} (-1)^{j_1-m_1} \begin{pmatrix} j_2 & J & j_1 \\ -m_2 & M & m_1 \end{pmatrix} (-1)^{j_3-m_3} \begin{pmatrix} j_3 & J & j_3 \\ -m_3 & M & m_3 \end{pmatrix}$$

Now, make use of the identity, easily obtained from Eq.(1.83),

$$\sqrt{2j_3+1}(-1)^{j_3-m_3} \begin{pmatrix} j_3 & 0 & j_3 \\ -m_3 & 0 & m_3 \end{pmatrix} = 1$$

to reexpress the sum over m_3 as

$$\begin{aligned} &\sum_{m_3} \begin{pmatrix} j_3 & J & j_3 \\ -m_3 & M & m_3 \end{pmatrix} \\ &= \sqrt{2j_3+1} \sum_{m_3} \begin{pmatrix} j_3 & J & j_3 \\ -m_3 & M & m_3 \end{pmatrix} (-1)^{j_3-m_3} \begin{pmatrix} j_3 & 0 & j_3 \\ -m_3 & 0 & m_3 \end{pmatrix} \\ &= \sqrt{2j_3+1}\, \delta_{J0}\, \delta_{M0} \end{aligned}$$

The sum over M therefore reduces to a single term, and the graph reduces to

$$\sqrt{2j_3+1}\, \delta_{J0}\, (-1)^{j_1-m_1} \begin{pmatrix} j_2 & 0 & j_1 \\ -m_2 & 0 & m_1 \end{pmatrix} = \sqrt{\frac{2j_3+1}{2j_1+1}} \delta_{J0}\, \delta j_1 j_2\, \delta_{m_1 m_2}$$