Zeeman effect: The vector potential for a uniform magnetic $\boldsymbol{B}$ can be written

$$\boldsymbol{A} = \frac{1}{2}[\boldsymbol{B} \times \boldsymbol{r}]$$

1. The interaction Hamiltonian is

$$h_{\rm int}(r) = -ec\, \boldsymbol{\alpha} \cdot \boldsymbol{A} = -\frac{ec}{2}\, \boldsymbol{\alpha} \cdot [\boldsymbol{B} \times \boldsymbol{r}] = -\frac{ec}{2}\, \boldsymbol{B} \cdot [\boldsymbol{r} \times \boldsymbol{\alpha}]$$

We may rewrite the cross product on the right side of this equation as

$$[\boldsymbol{r} \times \boldsymbol{\alpha}]_\lambda = -i\sqrt{2}\, r \left(\boldsymbol{\alpha} \cdot \boldsymbol{C}^{(0)}_{1\lambda}(\hat{r})\right)$$

Assuming $\boldsymbol{B}$ is in the z direction, the interaction becomes

$$h_{\rm int}(r) = i\frac{ecB}{2}\sqrt{2}\, r \left(\boldsymbol{\alpha} \cdot \boldsymbol{C}^{(0)}_{10}(\hat{r})\right)$$

2. The interaction energy W_v in a valence state $|v\rangle$ of a one electron atom may be written

$$W_v = \langle v|h_{\rm int}|v\rangle = i\frac{ecB}{2}\sqrt{2}\left\langle v \left| \left(\boldsymbol{\alpha} \cdot \boldsymbol{C}^{(0)}_{10}(\hat{r})\right) \right| v \right\rangle$$

The matrix element of $\boldsymbol{\alpha} \cdot \boldsymbol{C}^{(0)}_{10}$ above is

$$\begin{aligned} -i\int_0^\infty dr\, r\, P_v(r) Q_v(r) \Big[& \left\langle \kappa_v m_v \left| \left(\boldsymbol{\sigma} \cdot \boldsymbol{C}^{(0)}_{10}(\hat{r})\right) \right| -\kappa_v m_v \right\rangle \\ & - \left\langle -\kappa_v m_v \left| \left(\boldsymbol{\sigma} \cdot \boldsymbol{C}^{(0)}_{10}(\hat{r})\right) \right| \kappa_v m_v \right\rangle \Big] \\ & = i\left\langle -\kappa_v m_v \left| \left(\boldsymbol{\sigma} \cdot \boldsymbol{C}^{(0)}_{10}(\hat{r})\right) \right| \kappa_v m_v \right\rangle (r)_{vv} \end{aligned}$$

where

$$(r)_{vv} = 2\int_0^\infty dr\, r\, P_v(r) Q_v(r).$$

We note that

$$\left\langle -\kappa_v m_v \left| \left(\boldsymbol{\sigma} \cdot \boldsymbol{C}^{(0)}_{10}(\hat{r})\right) \right| \kappa_v m_v \right\rangle = \frac{2\kappa_v}{\sqrt{2}}\langle -\kappa_v m_v|C^1_0(\hat{r})|\kappa_v m_v\rangle.$$

Therefore, we have

$$W_v = \frac{ec}{2}\, B(-2\kappa_v)\langle -\kappa_v m_v|C^1_0(\hat{r})|\kappa_v m_v\rangle (r)_{vv}$$

The matrix element $\langle -\kappa_v m_v|C^1_0(\hat{r})|\kappa_v m_v\rangle$ evaluates to

$$\langle -\kappa_v m_v|C^1_0(\hat{r})|\kappa_v m_v\rangle = -\frac{m_v}{j(j+1)}.$$

3. In the Pauli approximation,

$$
\begin{aligned}
(r)_{vv} = & -\frac{1}{c}\int_0^\infty dr\, r\, P_v \left(\frac{dP_v}{dr} + \frac{\kappa_v}{r}P_v\right) \\
= & -\frac{1}{c}\int_0^\infty dr \left(\frac{1}{2}\frac{dr P_v^2}{dr} + \frac{1}{2}(2\kappa_v - 1)P_v^2\right) = -\frac{2\kappa_v - 1}{2c}
\end{aligned}
$$

Putting this together, we obtain the following expression for the interaction energy:

$$W_v = -\mu_B B\, g_v m_v, \tag{1}$$

where

$$g_v = \frac{\kappa_v(2\kappa_v - 1)}{j_v(j_v+1)} = \frac{2j_v + 1}{2l_v + 1}$$

is the Landé g-factor of the atomic term, and

$$\mu_B = \frac{e\hbar}{2m} \equiv \frac{e}{2}$$

is the Bohr magneton.

The factor g_v has the value 2, 2/3, 4/3, 4/5, 6/5, for $s_{1/2}$, $p_{1/2}$, $p_{3/2}$, $d_{3/2}$, $d_{5/2}$ states, respectively. In the above, $\mu_B = e/2m$ is the Bohr magneton. Its value is $e/2$ in atomic units.

Isotope Shift Answer to second problem is given in the mathematica notebook ProblemSet7.nb.