Zeeman effect: The vector potential for a uniform magnetic \( B \) can be written

\[
A = \frac{1}{2} (B \times r)
\]

1. The interaction Hamiltonian is

\[
h_{\text{int}}(r) = -\frac{e}{c} \alpha \cdot A = -\frac{e}{c} \frac{1}{2} \alpha \cdot [B \times r] = -\frac{e}{c} \frac{1}{2} B \cdot [r \times \alpha]
\]

We may rewrite the cross product on the right side of this equation as

\[
[r \times \alpha]_{\lambda} = -i\sqrt{2} r \left( \alpha \cdot C^{(0)}_{1\lambda}(\hat{r}) \right)
\]

Assuming \( B \) is in the \( z \) direction, the interaction becomes

\[
h_{\text{int}}(r) = -\frac{e}{c} B \frac{1}{2} \sqrt{2} r \left( \alpha \cdot C^{(0)}_{10}(\hat{r}) \right)
\]

2. The interaction energy \( W_v \) in a valence state \( |v\rangle \) of a one electron atom may be written

\[
W_v = \langle v | h_{\text{int}} | v \rangle = -\frac{e}{c} B \left( -\frac{1}{2} \kappa_v m_v \right) \langle \kappa_v m_v | \left( \sigma \cdot C^{(0)}_{10}(\hat{r}) \right) | \kappa_v m_v \rangle (r)_{vv}
\]

The matrix element of \( \alpha \cdot C^{(0)}_{10} \) above is

\[
- \int_0^{\infty} dr \, P_v(r) Q_v(r) \left[ \langle \kappa_v m_v | \left( \sigma \cdot C^{(0)}_{10}(\hat{r}) \right) \right] - \kappa_v m_v \rangle - \langle \kappa_v m_v | \left( \sigma \cdot C^{(0)}_{10}(\hat{r}) \right) | \kappa_v m_v \rangle \right]
\]

where

\[
(r)_{vv} = 2 \int_0^{\infty} dr \, P_v(r) Q_v(r).
\]

We note that

\[
\langle \kappa_v m_v | \left( \sigma \cdot C^{(0)}_{10}(\hat{r}) \right) | \kappa_v m_v \rangle = \frac{2\kappa_v}{\sqrt{2}} \langle \kappa_v m_v | C^{(0)}_{10}(\hat{r}) | \kappa_v m_v \rangle.
\]

Therefore, we have

\[
W_v = \frac{e}{c} B (2\kappa_v) \langle \kappa_v m_v | C^{(0)}_{10}(\hat{r}) | \kappa_v m_v \rangle (r)_{vv}
\]

The matrix element \( \langle \kappa_v m_v | C^{(0)}_{10}(\hat{r}) | \kappa_v m_v \rangle \) evaluates to

\[
\langle \kappa_v m_v | C^{(0)}_{10}(\hat{r}) | \kappa_v m_v \rangle = -\frac{m_v}{2}.
\]
3. In the Pauli approximation,

\[
(r)_{vv} = -\frac{1}{c} \int_0^\infty dr \ r P_v \left( \frac{dP_v}{dr} + \frac{\kappa_v}{r} P_v \right)
\]

\[
= -\frac{1}{c} \int_0^\infty dr \left( \frac{1}{2} \frac{dP_v^2}{dr} + \frac{1}{2} (2\kappa_v - 1) P_v^2 \right) = -\frac{2\kappa_v - 1}{2c}
\]

Putting this together, we obtain the following expression for the interaction energy:

\[
W_v = -\mu_B B g_v m_v, \quad (1)
\]

where

\[
g_v = \frac{\kappa_v (2\kappa_v - 1)}{j_v (j_v + 1)} = \frac{2j_v + 1}{2l_v + 1}
\]

is the Landé $g$-factor of the atomic term, and

\[
\mu_B = \frac{e\hbar}{2m} \equiv \frac{e}{2}
\]

is the Bohr magneton.

The factor $g_v$ has the value 2, 2/3, 4/3, 4/5, 6/5, for $s_{1/2}$, $p_{1/2}$, $p_{3/2}$, $d_{3/2}$, $d_{5/2}$ states, respectively. In the above, $\mu_B = e/2m$ is the Bohr magneton. Its value is $e/2$ in atomic units.

Isotope Shift  Answer to second problem is given in the mathematica notebook ProblemSet7.nb.