Work any three of the following four problems.

1. Derive the identity

\[
\begin{align*}
\sum_{j_1 m_1} & j_3 m_3 & = & \sum_{j_2 m_2} j_2 m_2 \\
\sum_{j_1 m_1} & j_3 m_3 & = & \sum_{j_2 m_2} j_2 m_2 \\
\end{align*}
\]

2. Show that the spherical harmonics \( Y_{kq}(\theta, \phi) \), \( q = -k, -k + 1, \ldots, k \) are components of a spherical tensor operator of rank \( k \).

\[
C(l, 1/2, j; m = m_s, m_s, m) = \begin{array}{c|cc}
m_s = 1/2 & m_s = -1/2 \\
j = l + 1/2 & \sqrt{\frac{1+m+1/2}{2l+1}} & \sqrt{\frac{l-m+1/2}{2l+1}} \\
j = l - 1/2 & -\sqrt{\frac{l-m+1/2}{2l+1}} & \sqrt{\frac{1+m+1/2}{2l+1}} \\
\end{array}
\]

3. Consider the 12-fold degenerate set of product wave functions:

\[
\psi_{2p,m_{\sigma}}(r_1)\psi_{1s,\mu}(r_2) = \frac{1}{r_1 r_2} P_{2p}(r_1) P_{1s}(r_2) Y_{1m}(\hat{r}_1) Y_{00}(\hat{r}_2) \chi_{\sigma}(1) \chi_{\mu}(2).
\]

(a) Combine these wave functions to give eigenstates of \( L^2, L_z, S^2, S_z \), where \( L = L_1 + L_2 \) and \( S = S_1 + S_2 \).

(b) With the aid of the above result, write down all possible antisymmetric angular momentum eigenstates describing 1s2p levels of helium. What is the number of such states?

4. Show that the ionization energy of an atom with one valence electron is \(-\epsilon_v\) in the “frozen-core” Hartree-Fock approximation. (Here, \( \epsilon_v \) is the eigenvalue of the valence electron Hartree-Fock equation.)