Your Answers to these problems will be collected in class on Sept. 7.

1. The ground state of hydrogen is a $1s$ state (orbital angular momentum eigenvalue $l = 0$). The total angular momentum (orbital + spin) of this state is $j = 1/2$ with two degenerate substates $m = \pm 1/2$. The angular wave functions of the two degenerate states are

$$|j, m\rangle = Y_{00}(\theta, \phi)\ \chi_m \equiv \sqrt{\frac{1}{4\pi}}\chi_m.$$

   Determine the values of total angular momentum of a $2p$ state (orbital angular momentum $l = 1$) of hydrogen and write out the angular wave functions for each of the 6 possible substates.

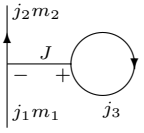
2. The hyperfine structure of an atom results from the interaction of bound atomic electrons with multipole moments of the nucleus. Determine the number of ground-state hyperfine levels of the hydrogen-like ion $^{209}\mathrm{Bi}^{+82}$. Find the eigenvalue f of total angular momentum $\boldsymbol{F} = \boldsymbol{I} + \boldsymbol{J}$ for each level and give its degeneracy.

   Hint: The nuclear spin of ^{209}Bi is $I = 9/2$. What is the corresponding value of the atomic angular momentum j in the ground state?

3. Prove [Varshalovich, Sec. 5.9.1, Eq.(5)]

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\ Y_{l_1,m_1}(\theta,\phi)\, Y_{l_2 m_2}(\theta,\phi)\, Y_{l_3 m_3}(\theta,\phi)$$
$$= \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

4. Derive the graphical identity



$$= \delta_{j_1 j_2}\ \delta_{m_1 m_2}\ \delta_{J0}\ \sqrt{\frac{2j_3+1}{2j_1+1}}$$