- Clebsch-Gordan Coefficients:

  1. Couple states $|j_1, m_1\rangle$ (eigenstates of J_{1z} and J_1^2) and states $|j_2, m_2\rangle$ (eigenstates of J_{2z} and J_2^2) to create a state $|j, m\rangle$ that is an eigenstate of J_1^2, J_2^2, J^2, and J_z, where $\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2$.

$$|j, m\rangle = \sum_{m_1, m_2} C(j_1, j_2, j; m_1, m_2, m)\ |j_1, m_1\rangle\ |j_3, m_2\rangle$$

  2. $C(j_1, j_2, j; m_1, m_2, m) = 0$ unless $m = m_1 + m_2$.
  3. Orthogonality:

$$\sum_{m_1, m_2} C(j_1, j_2, j; m_1, m_2, m) C(j_1, j_2, j'; m_1, m_2, m') = \delta j j' \delta_{mm'}$$

  4. Completeness:

$$\sum_{j,m} C(j_1, j_2, j; m_1, m_2, m) C(j_1, j_2, j; m_1', m_2', m) = \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

- Wigner 3-j symbols:

  1. Relation to CGC coefficients:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1 - j_2 - m_3}}{\sqrt{2j_3 + 1}}\ C(j_1, j_2, j_3; m_1, m_2, -m_3)$$

  2. Vanishes unless $m_1 + m_2 + m_3 = 0$
  3. Symmetry:
     (a) Symmetric under a cyclic permutation of columns.
     (b) Gains a phase $(-1)^{j_1+j_2+j_3}$ under an odd interchange of columns.
     (c) Gains a phase $(-1)^{j_1+j_2+j_3}$ under change of sign of all m's.
  4. Orthoganality & Completeness:

$$\sum_{m_1, m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \frac{\delta_{j_3 j_3'} \delta_{m_3 m_3'}}{2j_3 + 1}$$

$$\sum_{j_3, m_3} (2j_3 + 1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3 \end{pmatrix} = \delta_{m_1 n_1'} \delta_{m_2 m_2'}$$

- Irreducible Tensor Operators (of rank k): T^k_q, $q = -k,\ -k+1,\ \cdots\ k$

  1. Definition:

$$[J_z, T^k_q] = qT^k_q$$
$$[J_\pm, T^k_q] = \sqrt{(k \mp q)(k \pm q + 1)}\ T^k_{q\pm 1}$$

  2. Examples: $Y_{lm}(\theta.\phi)$ (a multiplicative operator of rank l) & J_μ (components of the angular momentum operator in a spherical basis, an operator of rank 1).
  3. Wigner-Ekhart Theorem:

$$\langle j_1, m_1 | T^k_q | j_2, m_2 \rangle = (-1)^{j_1 - m_1} \begin{pmatrix} j_1 & k & j_2 \\ -m_1 & q & m_2 \end{pmatrix} \langle j_1 \| T^k \| j_2 \rangle$$

  where $\langle j_1 \| T^k \| j_2 \rangle$ is independent of magnetic quantum numbers.
  4. $\langle j_1 \| J \| j_2 \rangle = \sqrt{j_1(j_1+1)(2j_1+1)}\ \delta_{j_1 j_2}$
  5. $C^l_m(\theta,\phi) = \sqrt{4\pi/(2l+1)}\ Y_{lm}(\theta,\phi)$

$$\langle l_1 \| C^l \| l_2 \rangle = \sqrt{(2l_1+1)(2l_2+1)}\ (-1)^{l_1} \begin{pmatrix} l_1 & k & l_2 \\ 0 & 0 & 0 \end{pmatrix}$$

- Graphical Rules: (See Lecture Notes for rules)

- Spherical Spinors

  1. $\Omega_{jlm}(\theta,\phi)$ are states obtained by coupling spherical harmonics and spin 1/2 states.

$$\Omega_{l+1lm}(\theta,\phi) = \begin{pmatrix} \sqrt{\frac{l+m+1/2}{2l+1}}\ Y_{l,m-1/2}(\theta,\phi) \\ \sqrt{\frac{l-m+1/2}{2l+1}}\ Y_{l,m+1/2}(\theta,\phi) \end{pmatrix}$$
$$\Omega_{l-1lm}(\theta,\phi) = \begin{pmatrix} -\sqrt{\frac{l-m+1/2}{2l+1}}\ Y_{l,m-1/2}(\theta,\phi) \\ \sqrt{\frac{l+m+1/2}{2l+1}}\ Y_{l,m+1/2}(\theta,\phi) \end{pmatrix}$$

  2. Ω_{jlm} are eigenstates of $K = -1 - \boldsymbol{\sigma}\cdot\boldsymbol{L}$ with eigenvalues $\kappa = \mp(j+1/2)$ for $j = l \pm 1/2$. Classify states by κ and use notation $\Omega_{\kappa m}$
  3. Orthogonality:

$$\int d\Omega\ \Omega^\dagger_{\kappa' m'} \Omega_{\kappa m} = \delta_{\kappa'\kappa}\delta_{m'm}$$

  4. Useful Identity: $\sigma \cdot \hat{r}\ \Omega_{\kappa m} = -\Omega_{-\kappa m}$

- Vector Spherical Harmonics: $Y_{JLM}(\theta, \phi)$ are states obtained by coupling spherical harmonics and spin 1 states.

  1. Orthogonality:

$$\int d\Omega\; Y^{\dagger}_{J'L'M'} Y_{JLM} = \delta_{J'J}\delta_{L'L}\delta_{M'M}$$

  2. Alternative Vector Spherical Harmonics

$$\begin{aligned} Y^{(-1)}_{JM} &= \left[\sqrt{\frac{J}{2J+1}} Y_{JJ-1M} - \sqrt{\frac{J+1}{2J+1}} Y_{JJ+1M}\right] \\ Y^{(1)}_{JM} &= \left[\sqrt{\frac{J+1}{2J+1}} Y_{JJ-1M} + \sqrt{\frac{J}{2J+1}} Y_{JJ+1M}\right] \\ Y^{(0)}_{JM} &= Y_{JJM} \end{aligned}$$

  3. Orthogonality:

$$\int d\Omega\; Y^{\dagger(\lambda')}_{J'M'}\; Y^{(\lambda)}_{JM} = \delta_{J'J}\;\delta_{\lambda'\lambda}\;\delta_{M'M}$$

  4. Useful Identities:

$$\begin{aligned} Y^{(-1)}_{JM} &= \hat{r}\; Y_{JM} \\ Y^{(1)}_{JM} &= \frac{r\nabla}{\sqrt{J(J+1)}}\; Y_{JM} \\ Y^{(0)}_{JM} &= \frac{L}{\sqrt{J(J+1)}}\; Y_{JM} \end{aligned}$$