Answers to Problem Set 1  Physics 607  (Sept. 3, 2001)

1. Derive the relations

\[ J^2 = J_+ J_- + J_+^2 - J_z, \]
\[ J^2 = J_- J_+ + J_-^2 + J_z. \]

**Solution:**

Consider the product \( J_+ J_- \)

\[ J_+ J_- = (J_x + iJ_y)(J_x - iJ_y) = J_x^2 + J_y^2 - i[J_x, J_y] = J_x^2 - J_y^2 + J_z, \]
\[ J_- J_+ = (J_x - iJ_y)(J_x + iJ_y) = J_x^2 + J_y^2 + i[J_x, J_y] = J_x^2 - J_y^2 - J_z. \]

where we have used \([J_x, J_y] = iJ_z\). Rearranging these expressions leads to

\[ J^2 = J_+ J_- + J_+^2 - J_z, \]
\[ J^2 = J_- J_+ + J_-^2 + J_z. \]

2. Show that the normalization factor \( c \) in the equation \( \Theta_{l,-l}(\theta) = c \sin^l \theta \)

\[ c = \frac{1}{2l!} \sqrt{\frac{(2l+1)!}{2}}, \]

and thereby verify that Eq. (1.30) is correct.

**Solution:**

\[ \int_{0}^{\pi} |\Theta_{l,-l}(\theta)|^2 \sin \theta d\theta = c^2 \int_{0}^{\pi} \sin^{2l+1}(\theta) d\theta = c^2 \int_{-1}^{1} (1 - x^2)^l dx = 1. \]

Evaluate \( I_l = \int_{-1}^{1} (1 - x^2)^l dx \) by parts:

\[ u = (1 - x^2)^l \quad dv = dx, \quad du = -2l x (1 - x^2)^{l-1} \quad v = x \]

From this, it follows

\[ I_l = x (1 - x^2)^l |_{-1}^{1} + 2l \int_{-1}^{1} x^2 (1 - x^2)^{l-1} dx \]
\[ = 2l \int_{-1}^{1} \left[ -(1 - x^2) + 1 \right] (1 - x^2)^{l-1} dx = 2l \left[ -I_l + I_{l-1} \right]. \]
This may be rewritten as the induction relation \( I_l = \frac{2l}{2l+1} I_{l-1} \).

Using the fact that \( I_0 = 2 \), one finds

\[
I_1 = \frac{2 \cdot 1}{3} 2
\]

\[
I_2 = \frac{2 \cdot 2 \cdot 1}{5 \cdot 3} 2
\]

\[
\ldots
\]

\[
I_l = \frac{2 \cdot l \cdots 2 \cdot 1}{(2l+1) \cdots 5 \cdot 3} 2 = \frac{2^l \cdot l!}{3 \cdot 5 \cdots (2l+1)} \cdot 2
\]

\[
= \frac{(2^l \cdot l!)^2}{(2l+1)!} 2
\]

It follows that

\[
c = \sqrt{\frac{1}{I_l}} = \frac{1}{2^l!} \sqrt{\frac{(2l+1)!}{2}}
\]

3. Write a MAPLE program to obtain the first 10 Legendre polynomials using Rodrigues’ formula.

Solution:

```maple
for l from 1 to 10 do
    P[l] := expand(diff((x^2-1)^l,x[l])/(2^l*l!));
end;
```

4. Legendre polynomials satisfy the recurrence relation

\[
lP_l(x) = (2l - 1)xP_{l-1}(x) - (l - 1)P_{l-2}(x).
\]

Write a MAPLE program to determine \( P_2(x), P_3(x), \ldots, P_{10}(x) \) (starting with \( P_0(x) = 1 \) and \( P_1(x) = x \)) using the above recurrence relation.

Solution:

```maple
P[0] := 1;
P[1] := x;
for l from 2 to 10 do
    P[l] := expand(((2*l-1)*x*P[l-1]-l*(l-1)*P[l-2])/l);
end;
```

5. Write a MAPLE program to generate the associated Legendre functions and \( P_l^m(x) \). Determine all \( P_l^m(x) \) with \( l \leq 4 \) and \( 1 \leq m \leq l \).
Solution:

for l from 1 to 4 do
    P[1,0] := expand(diff((x^2-1)^l,x^l)/(2^l*l!));
    for m from 1 to l do
        p[l,m] := (1-x^2)^(m/2) * expand(diff(P[1,0],x^m));
        print(P[l,m]= p[l,m]);
    od;
od;

P[1,0] := x  \quad P[1,1] = \sqrt{1-x^2}
P[2,0] := 3/2x^2 - 1/2  \quad P[2,1] = 3\sqrt{1-x^2}x  \quad P[2,2] = 3 - 3x^2
P[3,0] := 5/2x^3 - 3/2x \quad \ldots