Answers to Problem Set 1 Physics 607 (Sept. 3, 2001)

1. Derive the relations

$$
\begin{aligned}
J^2 &= J_+J_- + J_z^2 - J_z\,, \\
J^2 &= J_-J_+ + J_z^2 + J_z\,.
\end{aligned}
$$

Solution:
Consider the product J_+J_-

$$J_+J_- = (J_x + iJ_y)(J_x - iJy) = J_x^2 + J + y^2 - i[J_x, J_y] = J^2 - J_z^2 + J_z,$$
$$J_-J_+ = (J_x - iJ_y)(J_x + iJy) = J_x^2 + J + y^2 + i[J_x, J_y] = J^2 - J_z^2 - J_z,$$

where we have used $[J_x, J_y] = iJ_z$. Rearranging these expressions leads to

$$
\begin{aligned}
J^2 &= J_+J_- + J_z^2 - J_z\,, \\
J^2 &= J_-J_+ + J_z^2 + J_z\,.
\end{aligned}
$$

2. Show that the normalization factor c in the equation $\Theta_{l,-l}(\theta) = c\ \sin^l\theta$ is

$$c = \frac{1}{2^l l!}\sqrt{\frac{(2l+1)!}{2}},$$

and thereby verify that Eq. (1.30) is correct.

Solution:

$$\int_0^\pi |\Theta_{l,-l}(\theta)|^2 \sin\theta d\theta = c^2\int_0^\pi \sin^{2l+1}(\theta)d\theta = c^2\int_{-1}^1 (1-x^2)^l dx = 1.$$

Evaluate $I_l = \int_{-1}^1 (1-x^2)^l dx$ by parts:

$$u = (1-x^2)^l \quad dv = dx, \qquad du = -2l\,x(1-x^2)^{l-1} \quad v = x$$

From this, it follows

$$
\begin{aligned}
I_l &= x(1-x^2)^l\Big|_{-1}^1 + 2l\int_{-1}^1 x^2(1-x^2)^{l-1}dx \\
&= 2l\int_{-1}^1 \left[-(1-x^2)+1\right](1-x^2)^{l-1}dx = 2l\left[-I_l + I_{l-1}\right].
\end{aligned}
$$

This may be rewritten as the induction relation $I_l = 2l\, I_l/(2l+1)$. Using the fact that $I_0 = 2$, one finds

$$\begin{aligned}
I_1 &= \frac{2\cdot 1}{3}\, 2 \\
I_2 &= \frac{2\cdot 2\cdot 2\cdot 1}{5\cdot 3}\, 2 \\
\cdots &= \cdots \\
I_l &= \frac{2\cdot l\cdots 2\cdot 1}{(2l+1)\cdots 5\cdot 3}\, 2 = \frac{2^l\ l!}{3\cdot 5\cdots(2l+1)}\, 2 \\
&= \frac{\left(2^l\ l!\right)^2}{(2l+1)!}\, 2
\end{aligned}$$

It follows that

$$c = \sqrt{\frac{1}{I_l}} = \frac{1}{2^l l!}\sqrt{\frac{(2l+1)!}{2}}$$

3. Write a MAPLE program to obtain the first 10 Legendre polynomials using Rodrigues' formula.

   Solution:

```
for l from 1 to 10 do
   P[l] := expand(diff((x^2-1)^l,x$l)/(2^l*l!));
od;
```

4. Legendre polynomials satisfy the recurrence relation

$$lP_l(x) = (2l-1)xP_{l-1}(x) - (l-1)P_{l-2}(x).$$

   Write a MAPLE program to determine $P_2(x), P_3(x), \cdots, P_{10}(x)$ (starting with $P_0(x) = 1$ and $P_1(x) = x$) using the above recurrence relation.

   Solution:

```
P[0] := 1;
P[1] := x;
for l from 2 to 10 do
   P[l] := expand(((2*l-1)*x*P[l-1]-(l-1)P[l-1])/l);
od;
```

5. Write a MAPLE program to generate the associated Legendre functions and $P_l^m(x)$. Determine all $P_l^m(x)$ with $l \le 4$ and $1 \le m \le l$.

Solution:

```
for l from 1 to 4 do
P[l,0] := expand(diff((x^2-1)^l,x$l)/(2^l*l!));
   for m from 1 to l do
     p[l,m] := (1-x^2)^(m/2) * expand(diff(P[l,0],x$m));
     print(P[l,m]= p[l,m]);
   od;
od;
```

$P[1,0] := x$	$P[1,1] = \sqrt{1-x^2}$	
$P[2,0] := 3/2x^2 - 1/2$	$P[2,1] = 3\sqrt{1-x^2}\,x$	$P[2,2] = 3 - 3x^2$
$P[3,0] := 5/2x^3 - 3/2x$	$\cdots$	