Answers to Problem Set 2 Physics 607 (Sept. 10, 2001)

1. Program to test orthogonality relations:

```
read ‘cgc.map‘;
j1:= 1;
j2:=3/2;
jmin:=abs(j1-j2);
jmax:=j1+j2;
for J from jmin to jmax do
for M from -J to J do
for K from jmin to jmax do
for N from -K to K do
sp := 0;
for m1 from -j1 to j1 do
for m2 from -j2 to j2 do
sp := sp + cgc(j1,j2,J,m1,m2,M)*cgc(j1,j2,K,m1,m2,N);
od;
od;
if simplify(sp) <> 0 then print(ScPr(J,K,M,N) = simplify(sp)) fi;
od;
od;
od;
od;
```

Result:

$$\mathrm{ScPr}(1/2,1/2,-1/2,-1/2)=1$$
$$\mathrm{ScPr}(1/2,1/2,1/2,1/2)=1$$
$$\mathrm{ScPr}(3/2,3/2,-3/2,-3/2)=1$$
$$\mathrm{ScPr}(3/2,3/2,-1/2,-1/2)=1$$
$$\mathrm{ScPr}(3/2,3/2,1/2,1/2)=1$$
$$\mathrm{ScPr}(3/2,3/2,3/2,3/2)=1$$
$$\mathrm{ScPr}(5/2,5/2,-5/2,-5/2)=1$$
$$\mathrm{ScPr}(5/2,5/2,-3/2,-3/2)=1$$
$$\mathrm{ScPr}(5/2,5/2,-1/2,-1/2)=1$$
$$\mathrm{ScPr}(5/2,5/2,1/2,1/2)=1$$
$$\mathrm{ScPr}(5/2,5/2,3/2,3/2)=1$$
$$\mathrm{ScPr}(5/2,5/2,5/2,5/2)=1$$

The program to test the second orthogonality relation is similar.

2. Reduced matrix element of a normalized spherical harmonic:

$$\begin{aligned}\langle l_1||C^k||l_2\rangle &= (-1)^{l_1}\sqrt{(2l_1+1)(2l_2+1)}\begin{pmatrix} l_1 & k & l_2 \\ 0 & 0 & 0 \end{pmatrix} \\ &= (-1)^{l_1}\sqrt{(2l_1+1)(2l_2+1)}\begin{pmatrix} l_2 & k & l_1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= (-1)^{l_1-l_2}(-1)^{l_2}\sqrt{(2l_1+1)(2l_2+1)}\begin{pmatrix} l_2 & k & l_1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= (-1)^{l_1-l_2}\langle l_2||C^k||l_1\rangle\end{aligned}$$

The factor $(-1)^{l_1+k+l_2}$ from the interchange of l_1 and l_2 on the second line is just 1, since l_1+l_2+k is even.

3. Consider $[J_x, \sigma\cdot\hat{r}]$ We find:

$$\begin{aligned}[J_x, \sigma\cdot\hat{r}] &= \sigma_x[L_x, \frac{x}{r}] + \sigma_y[L_x, \frac{y}{r}] + \sigma_z[L_x, \frac{z}{r}] \\ &+ \frac{1}{2}\frac{x}{r}[\sigma_x, \sigma_x] + \frac{1}{2}\frac{y}{r}[\sigma_x, \sigma_y] + \frac{1}{2}\frac{z}{r}[\sigma_x, \sigma_z] \\ &= 0 + i\sigma_y\frac{z}{r} - i\sigma_z\frac{y}{r} \\ &+ 0 + i\frac{y}{r}\sigma_z - i\frac{z}{r}\sigma_y \\ &= 0.\end{aligned}$$

By the structute of the above relations, it follows that $[J_k, \sigma\cdot\hat{r}] = 0$ for each component J_k of $\boldsymbol{J}$; in particular for J_z. To demonstrate the commutation relation for J^2, we use the identity:

$$[J^2, \sigma\cdot\hat{r}] = \boldsymbol{J}\cdot[\boldsymbol{J}, \sigma\cdot\hat{r}] + [\boldsymbol{J}, \sigma\cdot\hat{r}]\cdot\boldsymbol{J} = 0 + 0 = 0$$

4.

$$
\begin{aligned}
&\frac{\boldsymbol{L}}{\sqrt{J(J+1)}} Y_{JM} \\
&\quad = \left(-\frac{1}{\sqrt{2}}(L_x - iL_y)\ \xi_{+1} + L_0\ \xi_0 + \frac{1}{\sqrt{2}}(L_x + iL_y)\ \xi_{-1} \right) \frac{Y_{JM}}{\sqrt{J(J+1)}} \\
&\quad = \left(-\sqrt{\frac{(J-M+1)(J+M)}{2J(J+1)}}\ \xi_{+1} Y_{JM-1} + \frac{M}{\sqrt{J(J+1)}}\ \xi_0 Y_{JM} \right. \\
&\qquad\qquad\qquad\qquad \left. + \sqrt{\frac{(J+M+1)(J-M)}{2J(J+1)}}\ \xi_{-1} Y_{JM+1} \right) \\
&\quad = C(J1J, M-1, 1, M)\xi_{+1} Y_{JM-1} + C(J1J, M, 0, M)\xi_0 Y_{JM} \\
&\qquad\qquad + C(J1J, M+1, -11, M)\xi_{-1} Y_{JM+1} \\
&\quad = Y_{JJM}
\end{aligned}
$$