1. Number of substates:

(a) \((nsnl)\) in \(LS\) coupling:

In this case, \(L = l\) and there are only two possibilities \(^1L\) and \(^3L\).
The first has \(2L + 1\) substates and the second has \((2S + 1)(2L + 1) = 3(2L + 1)\) substates. The total is \(N = 4(2L + 1) \equiv 8l + 4\) substates.

(b) \((nsnl)\) in \(jj\) coupling:

There are two possibilities \((ns_{1/2}n'l_{-1/2})\) with \(2 \times (2(l-1/2) + 1) = 4l\) substates and \((ns_{1/2}n'l_{+1/2})\) with \(2 \times (2(l + 1/2) + 1) = 4l + 4\) substates. The total is again \(N = 8L + 4\) substates.

(c) \((nd)\) in \(LS\) coupling:

Since \(L + S\) is even we have \(^1S\), \(^3P\), \(^1D\), \(^3F\), and \(^1G\). The number of states is \(N = 1 + 3 \times 3 + 5 + 3 \times 7 + 9 = 45\).

(d) \((nd)\) in \(jj\) coupling:

We have the \(jj\) configurations \((nd_{3/2})^2\), \((nd_{5/2})^2\), and \((nd_{3/2}nd_{5/2})\).
For \((nd_{3/2})^2\) only \(J = 0\) and \(2\) are possible with \(1 + 5 = 6\) substates; for \((nd_{5/2})^2\) only \(J = 0\), \(2\), and \(4\) are possible with \(1 + 5 + 9 = 15\) substates; for \((nd_{3/2}nd_{5/2})\), \(J = 1\), \(2\), \(3\), and \(4\) are possible with \(3 + 5 + 7 + 9 = 24\) substates. The total number of substates is thus \(N = 6 + 15 + 24 = 45\).

2. Cases (a-d) are normally ordered and have core expectation values 0. Cases (e) and (f) are not normally ordered. The core expectation values for case (e):

\[
\langle a^+_za^+_j b^+_a a^+_b \rangle = \delta_{bc} \delta_{ad} - \delta_{bd} \delta_{ac},
\]

and for case (f):

\[
\langle a^+_jb^+_j a^+_a c^+_c \rangle = \delta_{bc} \delta_{dc}.
\]

3. We have \(|I\rangle = a^+_a |0_c\rangle\) and \(|F\rangle = a^+_m a^+_b a^+_c |0_c\rangle\). It follows that

\[
\langle F|V_I|I\rangle = \frac{1}{2} \sum_{ijkl} g_{ijkl} (0_c |a^+_za^+_j b^+_a a^+_b : a^+_i a^+_j a^+_k : a^+_m : a^+_a : |0_c\rangle
\]

\[=
\frac{1}{2} \sum_{ijkl} g_{ijkl} (0_c : a^+_za^+_j b^+_a : a^+_i a^+_j a^+_k : a^+_m : a^+_a : |0_c\rangle
\]

which, using Wick’s theorem, reduces to

\[=
\frac{1}{2} \sum_{ijkl} g_{ijkl} (\delta_{ia} \delta_{jm} - \delta_{im} \delta_{ja}) (\delta_{kb} \delta_{lc} - \delta_{kb} \delta_{lc})
\]

\[= g_{ambc} - g_{amcb} = \tilde{g}_{ambc}
\]
4. The first-order energy is

\[ E^{(1)}(j_v,j_w,J) = \sum_{m's} \left( j_v m_v - j_w m_w \right) - \frac{JM}{j_a m_a} \langle 0_c | a^\dagger_w V_I a^\dagger_v a_a | 0_c \rangle , \]

where \( w \) & \( v \) and \( a \) & \( b \) differ only in the values of the magnetic quantum numbers. With the aid of Wick’s theorem, we find

\[ \langle 0_c | a^\dagger_b a_w V_I a^\dagger_a a_a | 0_c \rangle = g_{wabv} - g_{wvba} + \Delta_{wv} \delta_{ba} - \Delta_{ba} \delta_{wv} \]

The two-particle part of the first-order energy is

\[ \sum_{m's} \left( j_v m_v - j_w m_w \right) - \frac{JM}{j_a m_a} \sum_k \left[ \left( j_w m_w - j_a m_a \right) + X_k(wabv) \right] \]

After summing over \( m \) values, this reduces to

\[ \frac{(-1)^{J+j_v-j_a}}{|J|} \left[ X_J(wabv) + [J] \sum_k \left\{ \begin{array}{ccc} j_v & j_a & J \\ j_b & j_w & k \end{array} \right\} X_k(wabv) \right] . \]

This leads to the answer on setting \( w = v \) and \( b = a \).