1. Show that the state

\[ |J, M \rangle = \sum_{m_1 m_2} ^{j_1 m_1} \begin{pmatrix} J^M \end{pmatrix} |j_1, m_1 \rangle |j_2, m_2 \rangle , \]

is an eigenstate of \( \mathbf{J}_1 \cdot \mathbf{J}_2 \). What is the corresponding eigenvalue.

2. The interaction Hamiltonian for a one-electron atom in an external magnetic field directed along the \( z \)-axis is \( H_I = -\mu_0 \sigma_z B \).

(a) Express the energy shift of the state \( |n, j, j \rangle \) in terms of a reduced matrix element of \( \sigma \).

(b) Show that the energy shift of the states \( |n, j, m \rangle \) and \( |n, j, j \rangle \) are related by

\[ \Delta E_{njm} = \frac{m}{j} \Delta E_{nj j} . \]

Note:

\[ \left( \begin{array}{ccc} j & 1 & j \\ -m & 0 & m \end{array} \right) = (-1)^{j-m} \frac{m}{\sqrt{j(j+1)(2j+1)}} . \]

3. Hartree-Fock:

(a) Write down in detail the Hartree-Fock equations for the 3 closed subshells of the neon atom. (Give numerical values for the occupation numbers and exchange factors).

(b) Write down in detail the Dirac-Fock equations for the 4 closed subshells of the neon atom. (Give numerical values for the occupation numbers and exchange factors).

4. Compile and run the program \texttt{nrhf.f} for Li using the data set \texttt{li.in} as input. How do the HF eigenvalues compare with experiment? Modify the data set to determine the 2\( p \) and 3\( s \) energies for boron, assuming a "frozen" Be-like core and a single valence electron. How do these energies compare with experiment. (A description of the input data file is given at the beginning of the program)