(9.1) Check that \( f_1, f_2, \) and \( f_3 \) on pg. 365 of the text satisfy the wave equation and that \( f_4 \) and \( f_5 \) do not. Try this using MAPLE; the solution is elementary. For example,

\[
> f1 := \exp(-b*(x-v*t)^2);
\]

\[
f1 := e^{-b(x-v*t)^2}
\]

\[
> fxx := \text{diff}(f1,x,x);
\]

\[
fxx := -2b e^{(-b(x-v*t)^2)} + 4b^2 (x-v*t)^2 e^{(-b(x-v*t)^2)}
\]

\[
> ftt := \text{diff}(f1,t,t);
\]

\[
ftt := -2b v^2 e^{(-b(x-v*t)^2)} + 4b^2 (x-v*t)^2 v^2 e^{(-b(x-v*t)^2)}
\]

\[
> \text{simplify}(fxx-ftt/v^2);
\]

0

(9.2) Show that \( f(z, t) = A \sin k z \cos kvt \) satisfies the wave equation:

\[
\frac{\partial^2 f}{\partial z^2} - \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} = -k^2 A \sin k z \cos kvt + \frac{1}{v^2} (vk)^2 A \sin k z \cos kvt = 0.
\]

Express \( f(z, t) \) as the sum of two traveling waves:

\[
f(z, t) = \frac{1}{2} \left[ \sin k(z-vt) + \sin k(z+vt) \right].
\]

(9.3) Find \( A_3 \) and \( \delta_3 \) in Eq. (9.19):

\[
A_3 e^{i\delta_3} = A_1 e^{i\delta_1} + A_2 e^{i\delta_2}
\]

Taking real and imaginary parts of this expression one obtains:

\[
A_3 \cos \delta_3 = A_1 \cos \delta_1 + A_2 \cos \delta_2
\]

\[
A_3 \sin \delta_3 = A_1 \sin \delta_1 + A_2 \sin \delta_2.
\]

Squaring and adding, one finds

\[
A_3 = \sqrt{A_1^2 + A_2^2 + 2A_1A_2 \cos (\delta_1 - \delta_2)};
\]

whereas, dividing the lower equation by the upper leads to

\[
\delta_3 = \arctan \left( \frac{A_1 \sin \delta_1 + A_2 \sin \delta_2}{A_1 \cos \delta_1 + A_2 \cos \delta_2} \right).
\]
(9.4) Obtain Eq. (9.20) from the wave equation by separation of variables. Writing \( f(z, t) = Z(z)T(t) \), substituting into the wave equation, and dividing by \( Z(z)T(t) \), one obtains

\[
1 \frac{d^2 Z}{dz^2} = \frac{1}{v^2} \frac{d^2 T}{dt^2} = -k^2 ,
\]

where \( k \) is a separation constant. The resulting equations for \( Z \) and \( T \) are

\[
\frac{d^2 Z}{dz^2} + k^2 Z = 0 \\
\frac{d^2 T}{dt^2} + v^2 k^2 T = 0 .
\]

Setting \( \omega = v|k| \), we can write the solutions to these equations as

\[
Z(z) = e^{\pm ikz} \quad \text{and} \quad T(t) = e^{\pm i\omega t} .
\]

The corresponding solutions to the wave equation for right or left traveling waves are

\[
Ae^{i(kz-\omega t)} \quad \quad Be^{-i(kz+\omega t)}
\]

Superimposing waves with different amplitudes \( \tilde{A}(k) \), we obtain

\[
\tilde{f}(z, t) = \int_{-\infty}^{\infty} \tilde{A}(k) e^{i(kz-\omega t)} dk .
\]

(9.6) Motion of a string with a massive knot at \( z = 0 \).

(a) The boundary condition replacing Eq. (9.27) is obtained by evaluating the net vertical force from the string tension and equating it to the mass times acceleration of the knot:

\[
T \left( \frac{\partial f}{\partial z} \bigg|_{z=0+} - \frac{\partial f}{\partial z} \bigg|_{z=0-} \right) = m \frac{\partial^2 f}{\partial t^2} \bigg|_{z=0} = -m \omega^2 f(0, t) ,
\]

where the second term applies for a harmonic disturbance.

(b) Find the reflection coefficients \( A_R \) and the transmission coefficients \( A_T \) for the special case \( \mu_2 = 0 \). Note that \( v_2 = \infty \), which leads to \( k_2 = 0 \). Therefore, we have

\[
f(z, t) = e^{ik_1z-i\omega t} + A_R e^{-ik_1z-i\omega t} \quad \quad z \leq 0 \\
f(z, t) = A_T e^{-i\omega t} \quad \quad z \geq 0
\]

The boundary conditions lead to

\[
1 + A_R = A_T \\
-ik_1 T(1 - A_R) = -m \omega^2 A_T .
\]
Solving, we find

\[ A_T = \frac{2k_1}{k_1 - i(m/T)\omega^2} \]
\[ A_R = \frac{k_1 + i(m/T)\omega^2}{k_1 - i(m/T)\omega^2} \]

The magnitudes of these two coefficients are

\[ |A_T| = \frac{2k_1}{\sqrt{k_1^2 + (m^2/T^2)\omega^4}} \]
\[ |A_R| = 1, \]

and the phases are

\[ \delta_T = \arctan\left(\frac{m\omega^2}{k_1 T}\right) \]
\[ \delta_R = 2\delta_T. \]

(9.7) Viscous drag on a string:

(a) The equation of motion of the string obtained by balancing the mass of a string element times its acceleration to the sum of drag force and the force from the string tension on the element: (tension \( T \) and mass, length \( \mu \))

\[ \mu \Delta z \frac{\partial^2 f}{\partial t^2} = -\gamma \frac{\partial f}{\partial t} \Delta z + T \Delta z \frac{\partial^2 f}{\partial z^2}. \]

From this, follows the modified equation of motion:

\[ \mu \frac{\partial^2 f}{\partial t^2} + \gamma \frac{\partial f}{\partial t} - T \frac{\partial^2 f}{\partial z^2} = 0. \]

(b) Harmonic solution: \( f(z, t) = Z(z)e^{-i\omega t} \):

\[ \mu \omega^2 Z + i\omega \gamma Z + T \frac{\partial^2 Z}{\partial z^2} = 0, \]

or

\[ \frac{d^2 Z}{dz^2} + k^2 (1 + i\alpha) Z = 0, \]

where \( k = \omega/v \) with \( v = \sqrt{T/\mu} \), and where \( \alpha = \gamma/(\mu\omega) \). Seek solution in the form \( Ae^{i\eta z} \). One finds:

\[ \eta^2 = k^2 (1 + i\alpha) \]
(c) Setting \(a = \Re(\sqrt{1 + i\alpha})\) and \(b = \Im(\sqrt{1 + i\alpha})\), we may write the transmitted wave as:

\[ f(z, t) = Ae^{ikaz - kbz - i\omega t}, \]

which is an exponentially damped traveling wave. The 1/e falloff distance is

\[ z(1/e) = \frac{1}{bk}. \]

Note: we can write

\[ b = \Im(\sqrt{1 + i\alpha}) = \frac{\alpha}{2} \sqrt{\frac{2}{1 + \sqrt{1 + \alpha^2}}} = \frac{\alpha}{2} - \frac{\alpha^3}{16} + \frac{7\alpha^5}{256} \cdots. \]

Therefore, for small \(\alpha\),

\[ z(1/e) \approx \frac{2\mu \omega}{k\alpha} = \frac{2\mu \nu}{k\gamma}. \]

(d) Reflection coefficient: Following the standard procedure, we find

\[ A_R = \frac{k_1 - ak_2 - ibk_2}{k_1 + ak_2 + ibk_2} = |R|e^{i\delta_R}. \]

We obtain from this,

\[ |A_R| = \sqrt{\frac{(k_1 - ak_2)^2 + b^2k_2^2}{(k_1 + ak_2)^2 + b^2k_2^2}}, \]

and

\[ \delta_R = \arctan \left( \frac{bk_2}{k_1 - ak_2} \right) - \arctan \left( \frac{bk_2}{k_1 + ak_2} \right). \]

(9.9) Two traveling waves:

(a) Wave traveling in \(-\hat{x}\) direction and polarized in \(\hat{z}\) direction,

Complex form:

\[ E = \hat{z} E_0 e^{-ikx-i\omega t}, \quad B = -\frac{1}{c}[\hat{x} \times \hat{z}]E_0 e^{-ikx-i\omega t} = \frac{1}{c}\hat{y} E_0 e^{-ikx-i\omega t}. \]

Real form:

\[ E = \hat{z} E_0 \cos (kx + \omega t), \quad B = \frac{1}{c}\hat{y} E_0 \cos (kx + \omega t). \]
(b) Wave traveling in direction (1,1,1) and polarized in $x-z$ plane:

\[
\hat{k} = \frac{1}{\sqrt{3}}(\hat{x} + \hat{y} + \hat{z}) \\
\hat{n} = \frac{1}{\sqrt{2}}(\hat{x} - \hat{z}).
\]

The polarization vector $\hat{n}$ was chosen to be in $x-z$ plane and $\perp$ to $\hat{k}$. Note that

\[
\hat{p} \overset{\text{def}}{=} [\hat{k} \times \hat{n}] = \frac{1}{\sqrt{6}}(-\hat{x} + 2\hat{y} - \hat{z})
\]

The real form of the traveling wave is

\[
E = \hat{n}E_0 \cos (\mathbf{k} \cdot \mathbf{r} - \omega t) \\
B = \frac{1}{c} \hat{p}E_0 \cos (\mathbf{k} \cdot \mathbf{r} - \omega t)
\]

(9.10) The intensity of sunlight is $I = 1300 \text{w/m}^2$, what is the radiation pressure for absorbed and reflected light?

\[
P_{\text{abs}} = \frac{I}{c} = 433 \times 10^{-8} \text{N/m}^2 \\
P_{\text{refl}} = 2P_{\text{abs}} = 867 \times 10^{-8} \text{N/m}^2.
\]

What fraction of atmospheric pressure is this? Atmospheric pressure is $P_{\text{atm}} = 1.013 \times 10^5 \text{N/m}^2$. Therefore,

\[
P_{\text{abs}} = 4.27 \times 10^{-11} P_{\text{atm}} \\
P_{\text{refl}} = 8.55 \times 10^{-11} P_{\text{atm}}.
\]