1. A charge $Q$ is distributed uniformly along the $z$ axis from $z = -a$ to $z = a$. Write out the first few terms in the Legendre polynomials expansion of the potential for the case $r > a$.

Answer: The potential at a point $z > a$ on the axis may be written

$$\Phi(z) = \frac{1}{4\pi \epsilon_0} \frac{Q}{2a} \int_{-a}^{a} \frac{dz'}{z - z'} = \frac{1}{4\pi \epsilon_0} \frac{Q}{2a} \ln \left( \frac{z + a}{z - a} \right).$$

Expanding in powers of $a/z$, we find

$$\Phi(x) = \frac{1}{4\pi \epsilon_0} \frac{Q}{2a} \left[ \left( \frac{a}{z} - \frac{1}{2} \frac{a^2}{z^2} + \frac{1}{3} \frac{a^3}{z^3} + \cdots \right) \right.$$

$$\left. - \left( -\frac{a}{z} - \frac{1}{2} \frac{a^2}{z^2} - \frac{1}{3} \frac{a^3}{z^3} + \cdots \right) \right]$$

$$= \frac{1}{4\pi \epsilon_0} \frac{Q}{a} \left[ \frac{a}{z} + \frac{1}{3} \frac{a^3}{z^3} + \frac{1}{5} \frac{a^5}{z^5} + \cdots \right].$$

Replacing $1/z^{(l+1)}$ by $1/r^{(l+1)} P_l(\cos \theta)$, we obtain

$$\Phi(x) = \frac{Q}{4\pi \epsilon_0} \left[ \frac{1}{r} + \frac{1}{3} \frac{a^2}{r^3} P_2(\cos \theta) + \frac{1}{5} \frac{a^4}{r^5} P_4(\cos \theta) + \cdots \right].$$

Verify that your answer is correct for $r \gg a$.

Answer: For $r \gg a$, the above potential reduces to the potential of a point charge at the origin.

$$\Phi(x) \rightarrow \frac{Q}{4\pi \epsilon_0} \frac{1}{r}.$$

2. A point dipole $p$ is imbedded at the center of a dielectric sphere (radius $R$ and dielectric constant $\epsilon_r$). Find the potential inside and outside of the sphere.

Answer: Set up a boundary value problem: Assume that the potential has the form

$$\Phi^{\text{in}} = \frac{1}{4\pi \epsilon_0} \frac{p}{r^2} P_1(\cos \theta) + ArP_1(\cos \theta)$$

$$\Phi^{\text{out}} = \frac{B}{r^2} P_1(\cos \theta),$$

where $A$ and $B$ are expansion constants. Boundary conditions at $r = R$ lead to the two equations:

$$\frac{B}{R^2} = AR + \frac{p}{4\pi \epsilon_0} \frac{1}{R^2}$$

$$\frac{2B}{R^3} = \epsilon_r \left( -A + \frac{2p}{4\pi \epsilon_0} \frac{1}{R^3} \right).$$
Solving, we find

\[
A = \frac{2(\epsilon_r - 1)}{(\epsilon_r + 2)} \frac{p}{R^3} \frac{1}{4\pi \epsilon_0}
\]

\[
B = \frac{3\epsilon_r}{(\epsilon_r + 2)} \frac{p}{4\pi \epsilon_0}
\]

By what factor is the dipole moment \( p \) enhanced by the presence of the dielectric sphere?

Answer:

\[
p_{\text{out}} = \frac{3\epsilon_r}{(\epsilon_r + 2)} \ p
\]

3. A long cylinder of radius \( R \) has magnetization vector \( \mathbf{M} = k \rho^3 \hat{z} \), where \( k \) is a constant and \( \rho \) is the radius in cylindrical coordinates.

(a) Ignoring end effects, determine the bound current densities \( \mathbf{J}_b = \nabla \times \mathbf{M} \) and \( \mathbf{K}_b = \mathbf{M} \times \mathbf{n} \).

Answer:

\[
\mathbf{J}_b = \nabla \times \mathbf{M} = -3k \rho^2 \hat{\phi}
\]

\[
\mathbf{K}_b = \mathbf{M} \times \mathbf{n} = k R^3 \hat{\phi}
\]

(b) From the bound currents determine \( \mathbf{B} \) inside and outside the cylinder using Ampère’s law.

Answer: For \( \rho < R \) choose a rectangular loop; one side of the loop coincides with the axis, an opposite side is parallel to the axis at a distance \( \rho \) away. These two sides are connected by sides perpendicular to the axis. The total bound current through this loop is

\[
I_b = l \int_0^\rho 3k \rho^2 d\rho = lk \rho^3
\]

This current flows in the \(-\hat{\phi}\) direction. From Ampère’s law, taking into account the sense of the current, one finds that \( \mathbf{B} \) is in the +\( z \) direction and that

\[
B_z(\rho) = \mu_0 k \rho^3 = \mu_0 M(\rho), \quad \rho < a.
\]

If the above loop encloses the the surface of the cylinder, then the total enclosed current will be

\[
I_b = l \int_0^R 3k \rho^2 d\rho - lk R^3 = 0.
\]

and we will have

\[
B_z(\rho) = 0, \quad \rho < a.
\]
(c) Determine $H$ inside and outside the cylinder.
Answer: Outside $H = B/\mu_0 = 0$. Inside $H = B/\mu_0 - M = 0$. Therefore,
\[ H = 0, \text{ everywhere}. \]

4. An infinite straight wire carries a current
\[ I(t) = \left\{ \begin{array}{ll} 0 & t \leq 0 \\ I_0 & t > 0 \end{array} \right. \]
(a) Show that the (retarded) vector potential $A$ is in the $z$ direction and that
\[ A_z(\rho, t) = 0 \quad ct < \rho \\
= \frac{\mu_0 I_0}{2\pi} \ln \left( \frac{ct + \sqrt{c^2t^2 - \rho^2}}{\rho} \right) \quad ct \geq \rho. \]
Answer: For $ct < \rho$ no signal reaches $\rho$. For $ct > \rho$, only that segment of the wire with $|z| \leq \sqrt{c^2t^2 - \rho^2}$ contributes to the potential at $\rho$. For $ct > \rho$, we have
\[ A_z = \frac{\mu_0 I_0}{4\pi} \int_{-\sqrt{c^2t^2 - \rho^2}}^{\sqrt{c^2t^2 - \rho^2}} \frac{dz}{\sqrt{z^2 + \rho^2}} \\
= \frac{\mu_0 I_0}{4\pi} \ln \left( \frac{\sqrt{c^2t^2 - \rho^2} + ct}{-\sqrt{c^2t^2 - \rho^2} + ct} \right) \\
= \frac{\mu_0 I_0}{2\pi} \ln \left( \frac{ct + \sqrt{c^2t^2 - \rho^2}}{\rho} \right). \]

(b) Find $B(\rho, t)$ and $E(\rho, t)$. Show that your fields have the correct static limit as $t \to \infty$.
Answer: $E$ has only a $z$ component since
\[ E = -\frac{\partial A}{\partial t}. \]
We find
\[ E_z = \frac{\mu_0 I_0}{2\pi} \left( \frac{1}{ct + \sqrt{c^2t^2 - \rho^2}} \right) \left( c + \frac{c^2t}{\sqrt{c^2t^2 - \rho^2}} \right) \\
= \frac{\mu_0 I_0}{2\pi} \frac{c}{\sqrt{c^2t^2 - \rho^2}}. \]
As $t \to \infty$, $E_z \to 0$, as expected.
\[ B_\phi = -\frac{\partial A_z}{\partial \phi} \]
\[ = \frac{\mu_0 I_0}{2\pi} \left[ \frac{1}{\rho} + \left( \frac{1}{ct + \sqrt{c^2t^2 - \rho^2}} \right) \frac{\rho}{\sqrt{c^2t^2 - \rho^2}} \right] \]
\[ = \frac{\mu_0 I_0}{2\pi} \frac{ct}{\rho \sqrt{c^2t^2 - \rho^2}} \]
\[ \rightarrow \frac{\mu_0 I_0}{2\pi} \frac{1}{\rho} \text{ as } t \rightarrow \infty. \]

5. A transmission line consists of two long conducting ribbons of width \( w \), parallel to one another and separated by distance \( h \). A uniformly distributed current \( I \) runs up one conductor and back down the other as illustrated in the figure.

(a) Determine the direction and magnitude of the \( B \) field between the ribbons.

Answer: First note that the magnitude of the surface current density is \( K = \frac{I}{w} \). Using the coordinates shown in the figure, the \( B \) field between the ribbons is in the \(-z\) direction:

\[ B = -\mu_0 K \hat{z} = -\mu_0 \frac{I}{w} \hat{z}. \]

(b) Determine the magnetic energy stored in a section of length \( l \).

Answer:

\[ W_m = \frac{1}{2} \int d^3r \mathbf{B} \cdot \mathbf{H} = \frac{\mu_0 h I^2}{2w} l \]

(c) Find the self inductance per unit length of the transmission line.

Answer: Use \( W_m = (1/2)LI^2 \) to find

\[ L/l = \mu_0 \frac{h}{w}. \]

(d) Assuming that one end of the transmission line is connected to a power supply with terminal voltage \( V \) and the other end is terminated by a resistor \( R \), determine the direction and magnitude of Poynting vector. Express the magnitude in terms of \( V \) and \( R \).

Answer: Assuming the upper ribbon is at potential \( V \), the electric field is downward and has value

\[ \mathbf{E} = -\frac{V}{h} \hat{y} \]

The Poynting vector is

\[ \mathbf{S} = \mathbf{E} \times \mathbf{H} = \frac{V}{h} \frac{I}{w} \hat{x} = \frac{V^2}{RA} \hat{x}, \]
where \( A = wh \) is the area between the ribbons normal to the direction of energy flow (\( \hat{x} \)).

(e) Determine the direction and magnitude force/length exerted on the upper ribbon by the \( E \) field and by the \( B \) field.

Answer: The contribution to the force from the \( E \) field is

\[
dF_y = \varepsilon_0 \left[ E_y^2 - \frac{1}{2} E^2 \right] (-a) \, da = -\frac{\varepsilon_0}{2} \frac{V^2}{h^2} \, da,
\]

thus

\[
F^e_y / l = -\frac{\varepsilon_0 V^2 w}{2h^2}.
\]

The contribution from the \( B \) field is

\[
dF_y = \frac{1}{\mu_0} \left[ 0 - \frac{1}{2} B^2 \right] (-a) \, da = +\frac{\mu_0}{2} \frac{I^2}{w^2} \, da,
\]

and

\[
F^m_y / l = \frac{\mu_0 I^2}{2w}.
\]