1. A charge Q is distributed uniformly along the z axis from $z = -a$ to $z = a$. Write out the first few terms in the Legendre polynomials expansion of the potential for the case $r > a$.

   Answer: The potential at a point $z > a$ on the axis may be written

$$\Phi(z) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\int_{-a}^{a}\frac{dz'}{z-z'} = \frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\ln\left(\frac{z+a}{z-a}\right).$$

   Expanding in powers of a/z, we find

$$\begin{aligned}\Phi(x) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2a}\Bigg[&\left(\frac{a}{z} - \frac{1}{2}\left(\frac{a}{z}\right)^2 + \frac{1}{3}\left(\frac{a}{z}\right)^3 + \cdots\right)\\ &- \left(-\frac{a}{z} - \frac{1}{2}\left(\frac{a}{z}\right)^2 - \frac{1}{3}\left(\frac{a}{z}\right)^3 + \cdots\right)\Bigg]\\ &= \frac{1}{4\pi\epsilon_0}\frac{Q}{a}\left[\frac{a}{z} + \frac{1}{3}\left(\frac{a}{z}\right)^3 + \frac{1}{5}\left(\frac{a}{z}\right)^5 + \cdots\right]\end{aligned}$$

   Replacing $1/z^{(l+1)}$ by $1/r^{(l+1)}\,P_l(\cos\theta)$, we obtain

$$\Phi(x) = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r} + \frac{1}{3}\frac{a^2}{r^3}P_2(\cos\theta) + \frac{1}{5}\frac{a^4}{r^5}P_4(\cos\theta) + \cdots\right]$$

   Verify that your answer is correct for $r \gg a$.

   Answer: For $r \gg a$, the above potential reduces to the potential of a point charge at the origin.

$$\Phi(x) \to \frac{Q}{4\pi\epsilon_0}\frac{1}{r}.$$

2. A point dipole $\boldsymbol{p}$ is imbedded at the center of a dielectric sphere (radius R and dielectric constant ϵ_r). Find the potential inside and outside of the sphere.

   Answer: Set up a boundary value problem: Assume that the potential has the form

$$\begin{aligned}\Phi^{\text{in}} &= \frac{1}{4\pi\epsilon_0}\frac{p}{r^2}\,P_1(\cos\theta) + ArP_1(\cos\theta)\\ \Phi^{\text{out}} &= \frac{B}{r^2}\,P_1(\cos\theta),\end{aligned}$$

   where A and B are expansion constants. Boundary conditions at $r = R$ lead to the two equations:

$$\begin{aligned}\frac{B}{R^2} &= AR + \frac{p}{4\pi\epsilon_0}\frac{1}{R^2}\\ \frac{2B}{R^3} &= \epsilon_r\left(-A + \frac{2p}{4\pi\epsilon_0}\frac{1}{R^3}\right).\end{aligned}$$

Solving, we find

$$A = \frac{2(\epsilon_r - 1)}{(\epsilon_r + 2)} \frac{1}{R^3} \frac{p}{4\pi\epsilon_0}$$
$$B = \frac{3\epsilon_r}{(\epsilon_r + 2)} \frac{p}{4\pi\epsilon_0}$$

By what factor is the dipole moment $\boldsymbol{p}$ enhanced by the presence of the dielectric sphere?

Answer:

$$\boldsymbol{p}^{\text{out}} = \frac{3\epsilon_r}{(\epsilon_r + 2)} \boldsymbol{p}$$

3. A long cylinder of radius R has magnetization vector $\boldsymbol{M} = k\rho^3\hat{\boldsymbol{z}}$, where k is a constant and ρ is the radius in cylindrical coordinates.

   (a) Ignoring end effects, determine the bound current densities $\boldsymbol{J}_b = \boldsymbol{\nabla} \times \boldsymbol{M}$ and $\boldsymbol{K}_b = \boldsymbol{M} \times \boldsymbol{n}$.
   Answer:

$$\boldsymbol{J}_b = \boldsymbol{\nabla} \times \boldsymbol{M} = -\frac{dM}{d\rho}\hat{\boldsymbol{\phi}} = -3k\rho^2\hat{\boldsymbol{\phi}}$$
$$\boldsymbol{K}_b = \boldsymbol{M} \times \boldsymbol{n} = kR^3\hat{\boldsymbol{\phi}}$$

   (b) From the bound currents determine $\boldsymbol{B}$ inside and outside the cylinder using Ampère's law.
   Answer: For $\rho < R$ choose a rectangular loop; one side of the loop of length l coincides with the axis, an opposite side is parallel to the axis at a distance ρ away. These two sides are connected by sides perpendicular to the axis. The total bound current through this loop is

$$I_b = l \int_0^\rho 3k\rho^2 d\rho = lk\rho^3$$

   This current flows in the $-\hat{\boldsymbol{\phi}}$ direction. From Ampère's law, taking into account the sense of the current, one finds that $\boldsymbol{B}$ is in the $+z$ direction and that

$$B_z(\rho) = \mu_0 k\rho^3 = \mu_0 M(\rho), \qquad \rho < a.$$

   If the above loop encloses the the surface of the cylinder, then the total enclosed current will be

$$I_b = l \int_0^R 3k\rho^2 d\rho - lkR^3 = 0.$$

   and we will have

$$B_z(\rho) = 0, \qquad \rho < a.$$

(c) Determine $\boldsymbol{H}$ inside and outside the cylinder.
Answer: Outside $\boldsymbol{H} = \boldsymbol{B}/\mu_0 = 0$. Inside $\boldsymbol{H} = \boldsymbol{B}/\mu_0 - \boldsymbol{M} = 0$. Therefore,

$$\boldsymbol{H} = 0, \quad \text{everywhere.}$$

4. An infinite straight wire carries a current

$$I(t) = \begin{cases} 0 & t \leq 0 \\ I_0 & t > 0 \end{cases}$$

(a) Show that the (retarded) vector potential $\boldsymbol{A}$ is in the z direction and that

$$\begin{aligned} A_z(\rho, t) &= 0 & ct < \rho \\ &= \frac{\mu_0 I_0}{2\pi} \ln\left(\frac{ct + \sqrt{c^2t^2 - \rho^2}}{\rho}\right) & ct \geq \rho. \end{aligned}$$

Answer: For $ct < \rho$ no signal reaches ρ. For $ct > \rho$, only that segment of the wire with $|z| \leq \sqrt{c^2t^2 - \rho^2}$ contributes to the potential at ρ. For $ct > \rho$, we have

$$\begin{aligned} A_z &= \frac{\mu_0 I_0}{4\pi} \int_{-\sqrt{c^2t^2-\rho^2}}^{\sqrt{c^2t^2-\rho^2}} \frac{dz}{\sqrt{z^2 + \rho^2}} \\ &= \frac{\mu_0 I_0}{4\pi} \ln\left(\frac{\sqrt{c^2t^2 - \rho^2} + ct}{-\sqrt{c^2t^2 - \rho^2} + ct}\right) \\ &= \frac{\mu_0 I_0}{2\pi} \ln\left(\frac{ct + \sqrt{c^2t^2 - \rho^2}}{\rho}\right) \end{aligned}$$

(b) Find $\boldsymbol{B}(\rho, t)$ and $\boldsymbol{E}(\rho, t)$. Show that your fields have the correct static limit as $t \to \infty$.
Answer: $\boldsymbol{E}$ has only a z component since

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t}.$$

We find

$$\begin{aligned} E_z &= \frac{\mu_0 I_0}{2\pi} \left(\frac{1}{ct + \sqrt{c^2t^2 - \rho^2}}\right) \left(c + \frac{c^2 t}{\sqrt{c^2t^2 - \rho^2}}\right) \\ &= \frac{\mu_0 I_0}{2\pi} \frac{c}{\sqrt{c^2t^2 - \rho^2}}. \end{aligned}$$

As $t \to \infty$, $E_z \to 0$, as expected.

$\boldsymbol{B}$ has only a ϕ component and

$$\begin{aligned}
B_\phi &= -\frac{\partial A_z}{\partial \phi} \\
&= \frac{\mu_0 I_0}{2\pi}\left[\frac{1}{\rho} + \left(\frac{1}{ct+\sqrt{c^2t^2-\rho^2}}\right)\frac{\rho}{\sqrt{c^2t^2-\rho^2}}\right] \\
&= \frac{\mu_0 I_0}{2\pi}\frac{1}{\rho}\frac{ct}{\sqrt{c^2t^2-\rho^2}} \\
&\to \frac{\mu_0 I_0}{2\pi}\frac{1}{\rho} \quad \text{as } t \to \infty.
\end{aligned}$$

5. A transmission line consists of two long conducting ribbons of width w, parallel to one another and separated by distance h. A uniformly distributed current I runs up one conductor and back down the other as illustrated in the figure.

   (a) Determine the direction and magnitude of the B field between the ribbons.

   Answer: First note that the magnitude of the surface current density is $K = I/w$. Using the coordinates shown in the figure, the B field between the ribbons is in the $-z$ direction:

$$\boldsymbol{B} = -\mu_0 K\hat{\boldsymbol{z}} = -\mu_0\frac{I}{w}\hat{\boldsymbol{z}}.$$

   (b) Determine the magnetic energy stored in a section of length l.

   Answer:

$$W_m = \frac{1}{2}\int d^3r\boldsymbol{B}\cdot\boldsymbol{H} = \frac{\mu_0 h I^2}{2w}\,l$$

   (c) Find the self inductance per unit length of the transmission line.

   Answer: Use $W_m = (1/2)LI^2$ to find

$$L/l = \mu_0\frac{h}{w}.$$

   (d) Assuming that one end of the transmission line is connected to a power supply with terminal voltage V and the other end is terminated by a resistor R, determine the direction and magnitude of Poynting vector. Express the magnitude in terms of V and R.

   Answer: Assuming the upper ribbon is at potential V, the electric field is downward and has value

$$\boldsymbol{E} = -\frac{V}{h}\hat{\boldsymbol{y}}$$

   The Poynting vector is

$$\boldsymbol{S} = \boldsymbol{E}\times\boldsymbol{H} = \frac{V}{h}\frac{I}{w}\hat{\boldsymbol{x}} = \frac{V^2}{RA}\hat{\boldsymbol{x}},$$

where $A = wh$ is the area between the ribbons normal to the direction of energy flow ($\hat{\boldsymbol{x}}$).

(e) Determine the direction and magnitude force/length exerted on the upper ribbon by the E field and by the B field.

Answer: The contribution to the force from the E field is

$$dF_y = \epsilon_0 \left[E_y^2 - \frac{1}{2} E^2 \right] (-1)\, da = -\frac{\epsilon_0}{2} \frac{V^2}{h^2} da,$$

thus

$$F_y^e / l = -\frac{\epsilon_0 V^2 w}{2h^2}$$

The contribution from the B field is

$$dF_y = \frac{1}{\mu_0} \left[0 - \frac{1}{2} B^2 \right] (-1)\, da = +\frac{\mu_0}{2} \frac{I^2}{w^2} da,$$

and

$$F_y^m / l = \frac{\mu_0 I^2}{2w}.$$