1. Jackson 2.7: Green function for a plane.

(a) Green function: Let \( \mathbf{r} = (x, y, z) \) and \( \mathbf{r}' = (x', y', z') \), then

\[
G(\mathbf{r}, \mathbf{r}') = \frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} - \frac{1}{\sqrt{(x - x)^2 + (y - y)^2 + (z + z)^2}}
\]

(b) Solution for \( \Phi = V \) inside a circle of radius \( a \) on \( x - y \) plane. First we evaluate \( \frac{\partial G}{\partial n'} = -\frac{\partial G}{\partial x'} \bigg|_{z' = 0} = -\frac{2z}{\rho + \rho' + 2\rho\rho' \cos(\phi' - \phi)}^{3/2} \)

From azimuthal symmetry \( \Phi \) is independent of \( \phi \). It follows that

\[
\Phi(\rho, z) = \frac{zV}{2\pi} \int_0^a \int_0^{2\pi} \frac{d\phi'}{\left[z^2 + \rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi' - \phi)\right]^{3/2}}
\]

(c) For \( \rho = 0 \), we find

\[
\Phi(0, z) = \frac{zV}{2\pi} \int_0^a \int_0^{2\pi} \frac{d\phi'}{\left[z^2 + \rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi' - \phi)\right]^{3/2}} = \frac{zV}{2\pi} \int_0^a \frac{d\rho'^2}{\left[z^2 + \rho'^2\right]^{3/2}} = \frac{zV}{2\pi} \int_0^a \frac{d\rho'^2}{\left[z^2 + \rho'^2\right]^{3/2}} = V \left[1 - \frac{z}{\sqrt{z^2 + a^2}}\right]
\]

(d) Asymptotic expansion for \( z^2 + \rho^2 \gg a^2 \). Let \( x^2 = z^2 + \rho^2 \), then

\[
\frac{1}{\left[x^2 + \rho^2 - 2\rho\rho' \cos(\phi')\right]^{3/2}} = \frac{1}{x^3} \left[1 - \frac{3}{2} \frac{\rho^2}{x^2} + \frac{15}{8} \frac{(\rho^2 - 2\rho\rho' \cos(\phi'))^2}{x^4} + \cdots\right]
\]

\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{d\phi'}{\left[x^2 + \rho^2 - 2\rho\rho' \cos(\phi')\right]^{3/2}} = \frac{1}{x^3} \left[1 - \frac{3}{2} \frac{\rho^2}{x^2} + \frac{15}{8} \frac{2\rho^2 \rho'^2 + \rho'^4}{x^4} + \cdots\right]
\]

\[
\frac{1}{2\pi} \int_0^a \int_0^{2\pi} \frac{d\phi'}{\left[x^2 + \rho^2 - 2\rho\rho' \cos(\phi')\right]^{3/2}} = \frac{a^2}{2x^3} \left[1 - \frac{3}{4} \frac{a^2}{x^2} + \frac{5}{8} \frac{3\rho^2 a^2 + a^4}{x^4} + \cdots\right]
\]
Substituting $x^2 = z^2 + \rho^2$, we obtain

$$\Phi(\rho, z) = \frac{Va^2}{2(z^2 + \rho^2)^{3/2}} \left[ 1 - \frac{3}{4} \frac{a^2}{z^2 + \rho^2} + \frac{5}{8} \frac{3\rho^2a^2 + a^4}{(z^2 + \rho^2)^2} + \cdots \right]$$

On the axis, $\Phi(\rho, z)$ reduces to

$$\Phi(0, z) = \frac{Va^2}{2z^2} \left[ 1 - \frac{3}{4} \frac{a^2}{z^2} + \frac{5}{8} \frac{a^4}{z^4} + \cdots \right]$$

This is identical to the expansion for $z \gg a$ of the result from (c) above:

$$V \left( 1 - \frac{z}{\sqrt{z^2 + a^2}} \right) = \frac{Va^2}{2z^2} \left[ 1 - \frac{3}{4} \frac{a^2}{z^2} + \frac{5}{8} \frac{a^4}{z^4} + \cdots \right]$$

2. Jackson 2.11: Image potential for charged wire at $x = R$ parallel to a cylinder of radius $b$ centered at the origin.

(a) The potential in cylindrical coordinates is

$$\Phi(\rho, \phi) = \frac{1}{2\pi\epsilon_0} \left[ -\tau \ln \sqrt{\rho^2 + R^2} - 2R\rho\cos\phi \right.$$  
$$\left. + \tau' \ln \sqrt{\rho^2 + r^2} - 2r\rho\cos\phi \right]$$

where $r$ is the distance of the image from the axis of the cylinder. To achieve $\Phi = V$ on surface of the cylinder and $\lim_{\rho \to \infty} \Phi(\rho, \phi) = 0$, we choose $r = b^2/R$ and $\tau' = \tau$.

(b) With the above conditions, we find

$$\Phi(\rho, \phi) = \frac{\tau}{4\pi\epsilon_0} \ln \left[ \frac{\rho^2 + r^2 - 2r\rho\cos\phi}{\rho^2 + R^2 - 2R\rho\cos\phi} \right]$$

Note that the potential at the cylindrical surface is

$$V = \Phi(b, \phi) = \frac{\tau}{2\pi\epsilon_0} \ln \left[ \frac{b}{R} \right]$$

This equation relates the potential on the cylinder to the other parameters of the problem. For large $\rho$, we find

$$\Phi(\rho, \phi) = \frac{\tau}{2\pi\epsilon_0} \left[ \frac{R - r}{\rho} \cos\phi + \frac{R^2 - r^2}{2\rho^2} \cos2\phi + \frac{R^3 - r^3}{3\rho^3} \cos3\phi + \cdots \right]$$

(c) Induced charge density. The radial electric field at the surface is

$$E_\rho = -\frac{\partial \Phi}{\partial \rho} \bigg|_{\rho = b} = -\frac{\tau}{2\pi\epsilon_0 b} \frac{R^2 - b^2}{R^2 + b^2 - 2Rb\cos\phi}$$
Therefore
\[
\sigma = -\frac{\tau}{2\pi b} \frac{R^2 - b^2}{R^2 + b^2 - 2Rb\cos \phi}
\]

Below is a graph of the negative of the induced charge -\(\sigma\)

(d) The force on the charged wire: We first evaluate \(E_\rho\) at the wire.
\[
E_\rho(\rho, \phi) = -\frac{\tau}{2\pi \epsilon_0} \left[ \frac{\rho - r \cos \phi}{\rho^2 + r^2 - 2r \rho \cos \phi} - \frac{\rho - R \cos \phi}{\rho^2 + R^2 - 2R \rho \cos \phi} \right]
\]

From this, it follows that at the wire
\[
E_\rho(R, 0) = -\frac{\tau}{2\pi \epsilon_0} \frac{R - r}{R^2 + r^2 - 2rR} = -\frac{\tau}{2\pi \epsilon_0} \frac{1}{R - r}
\]

The force/length on the charged wire is, therefore,
\[
F_\rho/L = \tau E_\rho(R, 0) = -\frac{\tau^2}{2\pi \epsilon_0} \frac{1}{R - r} = -\frac{\tau^2}{2\pi \epsilon_0} \frac{R}{R^2 - b^2}
\]

3. **Jackson 2.13; “Cracking an interesting integral”**

(a) Let \(\Phi(b, \phi') = V_1\) for \(0 < \phi' < \pi\) and \(V_2\) for \(\pi < \phi' < 2\pi\). It follows from the Green function given in Prob. 2.12 that
\[
\Phi(\rho, \phi) = \frac{V_1}{2\pi} (1 - \xi^2) \int_0^\pi \frac{d\phi'}{1 + \xi^2 - 2\xi \cos (\phi' - \phi)} + \frac{V_2}{2\pi} (1 - \xi^2) \int_\pi^{2\pi} \frac{d\phi'}{1 + \xi^2 - 2\xi \cos (\phi' - \phi)}
\]

where \(\xi = \rho/b\). Notice that the second integral can be obtained from the first by the transformation \(\xi \rightarrow -\xi\). It follows that
\[
\Phi(\rho, \phi) = (1 - \xi^2) \frac{1}{2\pi} [V_1 I(\xi, \phi) + V_2 I(-\xi, \phi)]
\]

where
\[
I(\xi, \phi) = -\int_\phi^{\phi+\pi} \frac{d\psi}{1 + \xi^2 - 2\xi \cos \psi}
\]
With a change of variables to $x = e^{i\psi}$, we can then rewrite the integral as

$$I(\xi, \phi) = \frac{i}{\xi} \int_{-x_0}^{x_0} \frac{dx}{(x - \xi)(x - 1/\xi)}$$

where $x_0 = e^{i\phi}$. In this later form, the integral may be done using a partial fraction decomposition. One finds

$$I(\xi, \phi) = \frac{i}{1 - \xi^2} \ln \left[ \frac{1 - x_0 \xi (x_0^{-1} \xi + 1)}{1 + x_0 \xi (x_0^{-1} \xi - 1)} \right]$$

$$= \frac{i}{1 - \xi^2} \ln \left[ \frac{1 - \xi^2 - 2i\xi \sin \phi}{-1 + \xi^2 + 2i\xi \sin \phi} \right]$$

The absolute values of the numerator and denominator in the above fraction are equal. The phase of the numerator is

$$-\arctan(\xi \sin \phi/(1 - \xi^2))$$

and the phase of the denominator is

$$\pi + \arctan(\xi \sin \phi/(1 - \xi^2))$$

Therefore

$$I(\xi, \phi) = \frac{1}{1 - \xi^2} \left[ \pi + 2 \arctan \left( \frac{2\xi \sin \phi}{1 - \xi^2} \right) \right]$$

(n.b. This integral is given in Eq. (47a) on p. 100 of *Integraltafel, teil 2, Bestimmte Integrale*, W. Gröbner & N. Hofreiter, Springer (1958).)

Combining terms, we find

$$\Phi(\rho, \phi) = \frac{V_1}{2} + \frac{V_1 - V_2}{\pi} \arctan \left( \frac{2b \rho \sin \phi}{b^2 - \rho^2} \right)$$

(n.b. Jackson’s cylinder is rotated by $\pi/2$ with respect to our’s)

(b) Surface charge density:

$$\sigma = -\epsilon_0 E_r = \epsilon_0 \frac{\partial \Phi}{\partial \rho} \bigg|_{\rho=b} = \epsilon_0 \frac{V_1 - V_2}{\pi b} \csc \phi.$$

Equal and opposite charges accumulate on the two halves and the charge density diverges at the gap!

(c) Jackson 2.13; **Alternative solution.**

Expand $\Phi(\rho, \phi)$ in a series:

$$\Phi(\rho, \phi) = a_0 + \sum_{n=1}^{\infty} a_n \rho^n \cos n\phi + \sum_{n=1}^{\infty} b_n \rho^n \sin n\phi$$
The expansion coefficients are easily found in terms of the potentials on the surface.

\[ a_0 = \frac{V_1 + V_2}{2} \]

\[ a_n = 0, \quad n > 0 \]

\[ b_n = \frac{2(V_1 - V_2)}{b_n \pi} \quad n = 1, 3, \ldots \]

Therefore

\[ \Phi(\rho, \phi) = V_1 + V_2 - \frac{2(V_1 - V_2)}{b \pi} \sum_{m=1}^{\infty} \frac{1}{2m+1} \left( \frac{\rho}{b} \right)^{2m+1} \sin (2m+1) \phi \]

Now, let \( z = \rho e^{i\phi}/b \) and note that (twice) the sum becomes

\[ S = 2 \sum_{m=0}^{\infty} \frac{1}{2m+1} \left( \frac{\rho}{b} \right)^{2m+1} \sin (2m+1) \phi = -i \sum_{m=0}^{\infty} \frac{z^{2m+1}}{2m+1} - \text{c.c.} \]

Make a second transformation \( z = i\xi \) to obtain

\[ S = \left[ \sum_{m=0}^{\infty} \frac{(-1)^m \xi^{2m+1}}{2m+1} + \text{c.c.} \right] = \arctan \xi + \arctan \xi^* \]

From the rule \( \tan (A + B) = (\tan A + \tan B)/(1 - \tan A \tan B) \), it follows

\[ \tan S = \frac{\xi + \xi^*}{1 - \xi \xi^*} = -i \frac{z - z^*}{1 - zz^*} = 2 \frac{(\rho/b) \sin \phi}{1 - (\rho/b)^2} = 2 \frac{b \rho \sin \phi}{b^2 - \rho^2} \]

The sum \( S \) is therefore

\[ S = \arctan \left[ \frac{2b \rho \sin \phi}{b^2 - \rho^2} \right] \]

and

\[ \Phi(\rho, \phi) = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} \arctan \left( \frac{2b \rho \sin \phi}{b^2 - \rho^2} \right) \]

4. Jackson 2.16: Green Function for rectangular region. The potential is given by

\[ \Phi(x,y) = \frac{1}{4\pi \epsilon_0} \int_0^1 dx' \int_0^1 dy' G(x, y; x', y') \rho(x', y') \]

\[ = \frac{2}{\pi \epsilon_0} \sum_{n=1}^{\infty} \frac{\sin n\pi x}{n \sinh n\pi} \int_0^{\infty} \sin n\pi x' dx' \left[ \sinh n\pi (1 - y) \int_y^1 \sinh n\pi y' dy' \right] \rho(x', y') \]

\[ = \frac{2}{\pi \epsilon_0} \sum_{n=1}^{\infty} \frac{\sin n\pi y}{n \sinh n\pi} \int_0^{1-y'} \sinh n\pi (1 - y') dy' \rho(x', y') \]

\[ + \frac{2}{\pi \epsilon_0} \sum_{n=1}^{\infty} \frac{\sin n\pi y}{n \sinh n\pi} \int_y^1 \sinh n\pi y' dy' \rho(x', y') \]
For the case of a uniform charge distribution \( \rho(x, y) = 1 \), we can carry out the integrals easily:

\[
\int_0^1 \sin n\pi x' \, dx' = \frac{2}{n\pi} \text{ for odd } n \text{ and } 0 \text{ for even } n
\]

\[
\int_0^y \sinh n\pi y' \, dy' = \frac{1}{n\pi} \cosh n\pi y - 1
\]

\[
\int_y^1 \sinh n\pi (1 - y') \, dy' = \frac{1}{n\pi} \cosh n\pi (1 - y) - 1
\]

The combination arising from the \( y' \) integration can be simplified:

\[
(cosh n\pi y - 1) \sinh n\pi (1 - y) + (cosh n\pi (1 - y) - 1) \sinh n\pi y
\]

\[
= \sinh n\pi - \sinh n\pi y - \sinh n\pi (1 - y)
\]

\[
= \sinh n\pi \left[ 1 - \frac{2 \sinh \frac{n\pi}{2} \cosh n\pi (y - \frac{1}{2})}{\sinh n\pi} \right]
\]

\[
= \sinh n\pi \left[ 1 - \frac{\cosh n\pi (y - \frac{1}{2})}{\cosh \frac{n\pi}{2}} \right]
\]

Collecting terms and introducing \( m \) through \( n = 2m + 1 \), we obtain

\[
\Phi(x, y) = \frac{4}{\pi^3 \epsilon_0} \sum_{m=0}^{\infty} \frac{\sin (2m + 1)\pi x}{(2m + 1)^3} \left[ 1 - \frac{\cosh (2m + 1)\pi (y - \frac{1}{2})}{\cosh \frac{(2m + 1)\pi}{2}} \right]
\]
Here is a plot of $4\pi\epsilon_0\Phi(x, y)$ obtained by summing terms up to $m = 10$