1. Jackson Prob. 4.1: Multipole expansion for various charge distributions

(a) In the first case, we have 4 charges in the $xy$ plane at distance $a$ from the origin along the $\pm x$ and $\pm y$ axes.

\[
\Phi(r) = \frac{q}{4\pi\epsilon_0} \left[ \frac{1}{|r - a\hat{x}|} - \frac{1}{|r + a\hat{x}|} + \frac{1}{|r - a\hat{y}|} - \frac{1}{|r + a\hat{y}|} \right] 
\]

\[
= \frac{q}{4\pi\epsilon_0} \sum_l \frac{a^l}{r^{l+1}} \left[ P_l(\hat{r} \cdot \hat{x}) - P_l(-\hat{r} \cdot \hat{x}) + P_l(\hat{r} \cdot \hat{y}) - P_l(-\hat{r} \cdot \hat{y}) \right] 
\]

\[
= \frac{1}{\epsilon_0} \sum_{l=1,3,\ldots} \sum_m \frac{2qa^l}{r^{l+1}} \frac{1}{2l+1} Y_{lm}(\hat{r}) \left[ Y_{lm}^*(\hat{x}) + Y_{lm}^*(\hat{y}) \right] 
\]

Compare the last formula with the definition of $q_{lm}$ to find

\[
q_{lm} = 2qa^l \left[ Y_{lm}^*(\hat{x}) + Y_{lm}^*(\hat{y}) \right] 
\]

The polar angles of $\hat{x}$ and $\hat{y}$ are: $\theta_x = \theta_y = \pi/2$, $\phi_x = 0$, and $\phi_y = \pi/2$. Therefore,

\[
q_{11} = -(1 - i) \sqrt{\frac{3}{8\pi}} qa \\
q_{33} = -\frac{1}{4}(1 + i) \sqrt{\frac{35}{8\pi}} qa^3 \\
q_{31} = \frac{1}{4}(1 - i) \sqrt{\frac{21}{8\pi}} qa^3 \\
q_{55} = -\frac{3}{16}(1 - i) \sqrt{\frac{335}{8\pi}} qa^5 \\
q_{53} = \frac{1}{16}(1 + i) \sqrt{\frac{335}{8\pi}} qa^5 \\
q_{51} = -\frac{1}{8}(1 - i) \sqrt{\frac{165}{8\pi}} qa^5 
\]

All terms $q_{lm}$ with $l$ or $m$ odd vanish. Furthermore $q_{l-m} = (-1)^m q_{l,m}^*$.

Note: From text, one has

\[
q_{11} = -\sqrt{\frac{3}{8\pi}} (p_x - ip_y) \\
q_{10} = \sqrt{\frac{3}{4\pi}} p_z 
\]

It follows that $p = (2qa, 2qa, 0)$.

(b) In the second case we have charges $q$ at $z = \pm a$ balanced by a charge...
$-2q$ at the origin. We find

$$\Phi(r) = \frac{q}{4\pi\epsilon_0} \left[ \frac{1}{|r-a\hat{z}|} + \frac{1}{|r+a\hat{z}|} - \frac{2}{r} \right]$$

$$= \frac{q}{4\pi\epsilon_0} \sum_{l>0}^{a_l} \frac{\alpha_l}{r^{l+1}} [P_l(\hat{r} \cdot \hat{z}) + P_l(-\hat{r} \cdot \hat{x})]$$

$$= \frac{q}{4\pi\epsilon_0} \sum_{l=2,4,\ldots} \frac{2\alpha_l}{r^{l+1}} P_l(\hat{r} \cdot \hat{z})$$

$$= \frac{1}{\epsilon_0} \sum_{l=2,4,\ldots} \sum_{m} \frac{2\alpha_l}{r^{l+1}} \frac{1}{2l+1} Y_{lm}(\hat{r}) Y_{lm}^*(\hat{z})$$

Again compare with definition to find

$$q_{lm} = 2\alpha_l Y_{lm}^*(\hat{z})$$

As is well known

$$Y_{lm}^*(\hat{z}) = \sqrt{\frac{2l+1}{4\pi}} P_l(1) = \sqrt{\frac{2l+1}{4\pi}}$$

Therefore, for $l = 2, 4, \ldots$

$$q_{l0} = \sqrt{\frac{2l+1}{\pi}} \frac{\alpha_l}{q}, \quad q_{lm} = 0 \quad \text{for } m \neq 0.$$

The rectangular components of the quadrupole tensor are found by comparing with the formulas in text

$$Q = \begin{pmatrix} 4qa^2 & 0 & 0 \\ 0 & -2qa^2 & 0 \\ 0 & 0 & -2qa^2 \end{pmatrix}$$

(c) Plot the dominant contribution for second case as a function of $r$ in the $x - y$ plane.

$$\Phi(r, \mu) = \frac{1}{\epsilon_0} \frac{q_{l0}}{5^{l+1} \sqrt{4\pi}} P_2(\mu) = \frac{1}{4\pi\epsilon_0} \frac{2qa^2}{r^3} P_2(\mu)$$

The plot is shown below along with the plot required for the next item.

(d) Compare the plot required above with a plot of the exact potential. The two plots are shown together below. The distance $a$ is taken to be 1 in this case, and we plot $4\pi\epsilon_0\Phi(r)$. It can be seen that the quadrupole potential substantially overestimates the size of the potential (by 40% in this case) at $r = a$ but comes into close agreement as $r$ increases.
2. Jackson Prob. 4.6: Nucleus in a cylindrically symmetric field.

(a) Show that

\[ W = -\frac{1}{4} Q \left. \frac{\partial E_z}{\partial z} \right|_0 \]

In the principal axis system of a spheroidal nucleus, \( Q_{xx} = Q_{yy} = -\frac{Q_{zz}}{2} \). It follows that

\[ W = -\frac{1}{6} Q_{zz} \left[ \frac{\partial E_z}{\partial z} - \frac{1}{2} \frac{\partial E_x}{\partial x} - \frac{1}{2} \frac{\partial E_y}{\partial y} \right] \]

With the aid of

\[ \nabla \cdot \mathbf{E} = \frac{\partial E_z}{\partial z} + \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0 \]

One finds

\[ W = -\frac{1}{6} Q_{zz} \frac{3}{2} \frac{\partial E_z}{\partial z} = -\frac{1}{4} Q \left. \frac{\partial E_z}{\partial z} \right|_0 \]

(b) Given that \( W = 10 \text{ MHz} \) and \( Q = 10^{-28} \text{ m}^2 \), find the value of

\[ \frac{\partial E_z}{\partial z} = -\frac{4eW}{Q} \]

in units \( e/(4\pi\epsilon_0 a_0^3) \).

We obtain the following

\[ -\frac{4eW}{Q} = -8.27133 \times 10^{20} \text{ MKS} \]

\[ \frac{e}{4\pi\epsilon_0 a_0^3} = 9.71758 \times 10^{21} \text{ MKS} \]

\[ \frac{\partial E_z}{\partial z} = -0.085117 \frac{e}{4\pi\epsilon_0 a_0^3} \]
(c) Quadrupole moment of a uniformly charged spheroid with semimajor axis \(a\) and semiminor axis \(b\):

\[
Q = Q_{33} = 4\pi \rho_q \int_0^a dz \int_0^{\sqrt{1 - z^2/a^2}} \rho \rho(a^2 - \rho^2) = \frac{8\pi ab^2}{15} (a^2 - b^2) \rho_q,
\]

where

\[
\rho_q = q \frac{3}{4\pi ab^2}
\]

is the charge density. Therefore, in terms of the total charge \(q = Ze\),

\[
Q = \frac{2}{5}(a^2 - b^2) Ze = \frac{4}{5}(a + b) R Ze,
\]

where \(R = (a + b)/2\). It follows that

\[
\frac{(a - b)}{R} = \frac{5}{4} \frac{Q'}{R^2 Z} = \frac{1.25 \times 2.5 \times 10^{-28}}{(7 \times 10^{-15})^{2/63}} = 0.1012
\]

where \(Q' = Q/e\).

3. Jackson Prob. 4.8: A cylindrical shell (outer radius \(b\) - inner radius \(a\)) is filled with a material with dielectric constant \(\epsilon\) and placed in an electric field normal to it’s axis.

(a) Find the potential and electric field. We expand the potential in a series. In the outer region, \(r > b\), the potential takes the form

\[
\Phi(\rho, \phi) = \sum_n \left[ a_n \rho^n + \frac{b_n}{\rho^n} \right] \cos n\phi,
\]

where \(a_1 = -E_0\) and \(a_n = 0\), for \(n \neq 0\). (Also, \(b_0 = 0\).) As in the case of a dielectric sphere in an external field, only terms in the expansion with \(n = 1\) will be nonvanishing once the boundary conditions are applied. We therefore assume that the potential takes the form

\[
\Phi(\rho, \phi) = \begin{cases} 
  -E_0 \rho + \frac{c_1}{\rho} \cos \phi & b \leq \rho \\
  c_2 \rho + \frac{c_3}{\rho} \cos \phi & a \leq \rho \leq b \\
  c_4 \rho \cos \phi & 0 \leq \rho \leq a 
\end{cases}
\]

The four equations \(\Delta \Phi = 0\) at \(\rho = a, b\) and \(\Delta D = 0\) at \(\rho = a, b\) lead
to the following results for the expansion coefficients:

\[ c_1 = \frac{b^2 (a^2 - b^2) (\epsilon^2 - 1)}{a^2(\epsilon - 1)^2 - b^2(\epsilon + 1)^2} E_0 \]

\[ c_2 = -\frac{2b^2(\epsilon + 1)}{b^2(\epsilon + 1)^2 - a^2(\epsilon - 1)^2} E_0 \]

\[ c_3 = \frac{2a^2b^2(\epsilon - 1)}{a^2(\epsilon - 1)^2 - b^2(\epsilon + 1)^2} E_0 \]

\[ c_4 = \frac{4b^2\epsilon}{a^2(\epsilon - 1)^2 - b^2(\epsilon + 1)^2} E_0 \]

The electric field is as usual \( \mathbf{E} = -\nabla \Phi \). For the radial component, we have

\[ E_\rho(\rho, \phi) = \begin{cases} E_0 + \frac{c_1}{\rho^2} \cos \phi & b \leq \rho \\ -c_2 + \frac{c_3}{\rho^2} \cos \phi & a \leq \rho \leq b \\ -c_4 \cos \phi & 0 \leq \rho \leq a \end{cases} \]

For the angular component, we have

\[ E_\phi(\rho, \phi) = \begin{cases} -E_0 + \frac{c_1}{\rho^2} \sin \phi & b \leq \rho \\ c_2 + \frac{c_3}{\rho^2} \sin \phi & a \leq \rho \leq b \\ c_4 \sin \phi & 0 \leq \rho \leq a \end{cases} \]

(b) Sketch the Field: Here is the case \( \epsilon = 10, \ a = 1, \ b = 2, \ E_0 = 1 \)
(c) Limiting case $a \to 0$:

$$\Phi(\rho, \phi) = \begin{cases} 
-\rho + \frac{\epsilon - 1}{\epsilon + 1} \frac{b^2}{\rho} E_0 \cos \phi & b \leq \rho \\
-\frac{2}{\epsilon + 1} \rho E_0 \cos \phi & 0 \leq \rho \leq b
\end{cases}$$

For the case of a hollow cylinder imbedded in a dielectric the potential is given by the above expression with $\epsilon \to 1/\epsilon$.

4. Jackson Prob. 4.13: The energy of a dielectric material in an external electric field is given by Eq. (4.93) in the text:

$$W = -\frac{1}{2} \int P \cdot E \, d\tau.$$  

For the liquid in the capillary tube,

$$P = \epsilon_0 \chi_e E$$

Therefore

$$W = -\frac{\epsilon_0 \chi_e}{2} \int E^2 \, d\tau = -\frac{\epsilon_0 \chi_e}{2} \left( \frac{V}{\ln(b/a)} \right)^2 2\pi x \int_a^b \rho \, d\rho \, d\rho$$

where we have used the easily established fact that the electric field in the capillary is

$$E_\rho(\rho) = \frac{V}{\ln(b/a)} \frac{1}{\rho}.$$  

It follows that

$$W = -\frac{\pi \epsilon_0 \chi_e V^2}{\ln(b/a)} x$$

and that the (upward) force on the liquid is

$$F = -\frac{dW}{dx} = \frac{\pi \epsilon_0 \chi_e V^2}{\ln(b/a)}$$

This force balances the downward weight of column of liquid $\rho gh \pi (b^2 - a^2)$. Therefore,

$$\chi_e = \frac{\rho gh (b^2 - a^2) \ln(b/a)}{\epsilon_0 V^2}$$

Here is an alternative solution mentioned in class: Let the capacitance/length of the capillary tube be

$$C_0 = \frac{2\pi \epsilon_0}{\ln(b/a)}$$

and the capacitance/length of the tube filled with liquid be

$$C = \frac{2\pi \epsilon}{\ln(b/a)}$$
If the tube is filled to height $x$ with liquid, an excess charge

$$\Delta Q = (C - C_0)xV = \frac{2\pi \epsilon_0 \chi_e x V}{\ln(b/a)}$$

will be drawn from the battery and appear on the surface of the capillary. The battery gives up energy

$$W_B = V \Delta Q = (C - C_0)xV^2$$

Part of this energy is the increased energy stored in the capacitor part is available to do work. Assuming that the capillary is filled to height $h$, the available energy is

$$W_a(x) = \frac{1}{2}(C - C_0)xV^2 - W_B = -\frac{1}{2} \frac{2\pi \epsilon_0 \chi_e x V^2}{\ln(b/a)}$$

The force on the (liquid) dielectric is

$$F = \frac{dW_a}{dx} = \frac{\pi \epsilon_0 \chi_e x V^2}{\ln(b/a)},$$

which agrees with the previously obtained result.