1. Jackson Prob. 5.1: Reformulate the Biot-Savart law in terms of the solid angle subtended at the point of observation by the current-carrying circuit.

\[ B(r) = \frac{\mu_0 I}{2\pi} \oint \frac{dl' \times (r - r')}{|r - r'|^3} \]

\[ = -\frac{\mu_0 I}{2\pi} \oint \frac{dl'}{|r - r'|} \nabla \frac{1}{|r - r'|} \]

\[ = \frac{\mu_0 I}{2\pi} \nabla \times \oint \frac{dl'}{|r - r'|} \]

Let

\[ V = \oint \frac{dl'}{|r - r'|} \]

then

\[ B(r) = \frac{\mu_0 I}{2\pi} \nabla \times V \]

The ith component of \( V \) may be written

\[ V_i = \oint \frac{dl' \cdot \hat{i}}{|r - r'|} \]

where \( \hat{i} \) is the unit vector along the ith axis. By virtue of Stoke’s theorem this can be converted into a surface integral

\[ V_i = \int_S da \left[ \nabla' \times \left( \frac{\hat{i}}{|r - r'|} \right) \right] \cdot \mathbf{n}' = \int_S da \left[ \mathbf{n}' \times \nabla' \frac{1}{|r - r'|} \right] \cdot \hat{i} \]

where \( S \) is a surface bounded by the circuit and where the direction of the surface normal \( \mathbf{n}' \) is related to the sense of the current \( (l') \) by the right-hand rule. The above equation can be rewritten as

\[ V = \nabla \times \int_S da \frac{\mathbf{n}'}{|r - r'|} \]

Therefore

\[ B(r) = \frac{\mu_0 I}{4\pi} \nabla \times [\nabla \times W] \]

with

\[ W = \int_S da \frac{\mathbf{n}'}{|r - r'|} \]

Now

\[ \nabla \times [\nabla \times W] = \nabla (\nabla \cdot W) - \nabla^2 W \]

\[ = -\nabla \int_S da \frac{\mathbf{n}' \cdot (r - r')}{|r - r'|^3} + 4\pi \int_S da \mathbf{n}' \delta(r - r') \]

\[ = -\nabla \Omega + 0. \]
The second integral vanishes since \( r' \) is on a surface bounding the circuit, which is away from the observation point \( r \). The first integral is, as shown on page 33 in Chap. 1 of the text, the solid angle \( \Omega \) subtended at the observation point by the circuit that bounds \( S \). Therefore,

\[
B(r) = -\frac{\mu_0 I}{4\pi} \nabla \Omega
\]

**Example:** Consider a point of observation on the \( z \) axis above a circular loop of radius \( a \) in the \( xy \) plane that carries current \( I \). The loop subtends a solid angle \( \Omega = 2\pi(1 - \cos \theta) \), where \( \theta \) is the angle between the \( z \) axis and a line from the point of observation to the loop. Thus

\[
\Omega = 2\pi \left[ 1 - \frac{z}{\sqrt{a^2 + z^2}} \right]
\]

and

\[
\nabla \Omega = 2\pi \left[ -\frac{1}{\sqrt{a^2 + z^2}} + \frac{z^2}{(a^2 + z^2)^{3/2}} \right] \hat{z} = -\frac{2\pi a^2}{(a^2 + z^2)^{3/2}} \hat{z}.
\]

Therefore

\[
B(z) = \frac{\mu_0 I}{2} \frac{a^2}{(a^2 + z^2)^{3/2}} \hat{z},
\]

confirming a result obtained in class directly from the Biot-Savart law.

2. **Jackson Prob 5.3:** Find \( B_z \) inside a uniformly wound solenoid. Use result from Prob. 5.1 to write the contribution from a segment of the solenoid of length \( dz \) as

\[
dB_z = -\frac{\mu_0 N I dz}{4\pi} \frac{d\Omega}{dz} = -\frac{\mu_0 N I}{4\pi} d\Omega.
\]

where \( \Omega = 2\pi(1 - \cos \theta) \) where \( \theta \) is the angle that the line from the observation point to the ring at \( z \) makes with the axis. Integrate from end 1 to end 2 to find

\[
B_z = \frac{\mu_0 NI}{4\pi} [\Omega_1 - \Omega_2] = \frac{\mu_0 NI}{2} [\cos \theta_2 - \cos \theta_1].
\]

In terms of the angles shown in the figure in the text this becomes

\[
B_z = \frac{\mu_0 NI}{2} [\cos \theta_1 + \cos \theta_2]
\]

3. **Jackson Prob 5.7:**

(a) As shown in class and in example with Prob. 5.1, the field of a single loop in the \( xy \) plane at a distance \( z \) from its center on the axis is

\[
B_z = \frac{\mu_0 I}{2} \frac{a^2}{(a^2 + z^2)^{3/2}}
\]
(b) The field near the center of a Helmholtz pair is, therefore,

\[ B_z = \frac{\mu_0 I}{2} \left[ \frac{a^2}{|a^2 + (z - b/2)^2|^{3/2}} + \frac{a^2}{|a^2 + (z - b/2)^2|^{3/2}} \right] \]

\[ = \frac{\mu_0 Ia^2}{d^3} \left[ 1 + \frac{3 (b^2 - a^2) z^2}{2 d^4} + \frac{15 (2a^4 - 6b^2a^2 + b^4) z^4}{16 d^8} \right. \]

\[ - \frac{7 (5a^6 - 30b^2a^4 + 15b^4a^2 - b^6)}{16 d^{12}} z^6 + \ldots \],

where \( d = \sqrt{b^2 + 4a^2} \).

(c) \( \rho \) dependence of field. On the axis, we may write \( B_z = \sigma_0 + \sigma_2 z^2 + \ldots \), where the coefficients \( \sigma_k \) can be inferred from the above equation. With the aid of \( \nabla \cdot B = 0 \) we find that near the origin,

\[ \frac{\partial (\rho B_{\rho})}{\partial \rho} = -\rho \frac{\partial B_z}{\partial z} = -2\sigma_2 \rho z + \ldots \]

Solving for \( B_{\rho} \) (taking into account that \( B_{\rho} = 0 \) on axis) we find that near the axis,

\[ B_{\rho}(z, \rho) \approx -\sigma_2 \rho z \]

From \( \nabla \times B = 0 \), we may write

\[ \frac{\partial B_z}{\partial \rho} = \frac{\partial B_{\rho}}{\partial z} \approx -\sigma_2 \rho \]

Solving for \( B_z \), we obtain

\[ B_z(z, \rho) \approx B_z(z, 0) - \sigma_2 \rho^2/2 = \sigma_0 + \sigma_2 \left( z^2 - \frac{\rho^2}{2} \right) + \ldots \]

(d) From Mathematica the asymptotic series for \( B_z \) is

\[ B_z = \frac{\mu_0 Ia^2}{|z|^3} \left[ 1 + \frac{3 (b^2 - a^2)}{2 z^2} + \frac{15 (2a^4 - 6b^2a^2 + b^4)}{16 z^4} \right. \]

\[ - \frac{7 (5a^6 - 30b^2a^4 + 15b^4a^2 - b^6)}{16 z^6} + \ldots \],

which can be obtained from the power series by the replacement \( d \rightarrow |z| \).

(e) For a Helmholtz coil one sets \( b = a \). With this choice the terms in the bracket for the small \( z \) expansion become

\[ \ldots \approx 1 - \frac{144z^4}{125a^4} \]
Thus, to differ from uniformity by \( \leq \epsilon \), the fractional distance must satisfy \( \frac{z}{a} \leq \frac{1}{4} \left( \frac{125}{144} \right) \epsilon^{1/4} \). For \( \epsilon = 10^{-4} \) the limit is 0.097 and for \( \epsilon = 10^{-2} \) the limit is 0.305. Below is a figure showing the variation of \( B_z \) on the axis between two coils located at \( a = \pm 1 \).

4. Jackson Prob 5.13: Find the vector potential and magnetic induction for a uniformly charged sphere of radius \( a \) rotating about an axis with angular momentum \( \omega \). We orient \( \omega \) along the \( z \) axis and let \( r \) lie in the \( xz \) plane. The vector \( r' \) is used to locate a point on the sphere. The surface current density at \( r' \) is

\[
J(r') = \frac{\sigma}{4\pi} \omega \times r'.
\]

The corresponding vector potential is

\[
A(r) = \frac{\mu_0}{4\pi} \int \frac{J(r')d^3r'}{|r - r'|}.
\]

We write

\[
[\omega \times r'] = a\omega \sin \theta'(-\sin \phi' \hat{x} + \cos \phi' \hat{y})
\]

As in the example worked out in Sec. 5.5 of the text, only the \( y \) component can contribute to the integral. Therefore,

\[
A_y(r) = \frac{\mu_0 \sigma a^3 \omega}{4\pi} \int \frac{\sin \theta' \cos \phi' d\Omega'}{\sqrt{r^2 + a^2 - 2ar \cos \gamma}},
\]

where \( \cos \gamma \) is the angle between \( r \) and \( r' \). Expanding the denominator in a series of spherical harmonics we obtain

\[
A_y(r) = \frac{\mu_0 \sigma a^3 \omega}{4\pi} \sum_{lm} \frac{r_<^l}{2l + 1} Y^*_{lm}(\hat{r}) \int d\Omega' \sin \theta' \cos \phi' Y_{lm}(\hat{r}')
\]

We first carry out the \( \phi' \) integral to find

\[
\int_0^{2\pi} d\phi' Y_{lm}(\theta', \phi') = \pi \sqrt{\frac{(2l + 1)(l - 1)!}{4\pi(l + 1)!}} P^1_l(\cos \theta') (\delta_{m1} - \delta_{m-1}).
\]

Noting that \( \sin \theta' = -P^1_l(\cos \theta') \), we find

\[
\int_{-1}^1 \sin \theta' P^1_l(\mu') d\mu' = -\frac{2}{3} \delta_{l1}
\]
Putting the previous two results together, we find
\[
\int \, \Omega' \, \sin \theta' \, \cos \phi' \, Y_{lm}(\hat{r}') = -\frac{4\pi}{3} \sqrt{\frac{3}{8\pi}} \left( \delta_{m1} - \delta_{m-1} \right) \delta_{l1}
\]
The sum over \(lm\) above becomes
\[
\sum_{lm} \frac{r^l}{r^{l+1}} \cdots = -\frac{16\pi^2}{9} \sqrt{\frac{3}{8\pi}} \left( Y^*_{11}(\hat{r}) - Y^*_{1-1}(\hat{r}) \right) \frac{r^<}{r^>^2} = \frac{4\pi}{3} \sin \theta \frac{r^<}{r^>^2}
\]
Finally,
\[
A_\phi(r) = \begin{cases} 
\frac{\mu_0 \sigma a^4 \omega}{3} \sin \theta \frac{1}{r^2} & r > a \\
\frac{\mu_0 \sigma a \omega}{3} r \sin \theta & r < a 
\end{cases}
\]
In vector form, this becomes
\[
A(r) = \begin{cases} 
\frac{\mu_0 \sigma a^4}{3} \left[ \omega \times r \right] & r > a \\
\frac{\mu_0 \sigma a}{3} \left[ \omega \times r \right] & r < a 
\end{cases}
\]
The corresponding formulas for the magnetic induction \(B = \nabla \times A\) are
\[
B(r) = \begin{cases} 
\frac{\mu_0 \sigma a^4}{3} \left( 3(\omega \cdot \hat{r})\hat{r} - \omega \right) & r > a \\
\frac{2\mu_0 \sigma a}{3} - \omega & r < a 
\end{cases}
\]
Note: A somewhat different (and simpler) solution to this problem is found in the text by Griffiths. He chooses coordinates with \(r\) be along the \(z\) axis and \(\omega\) in the \(xz\) plane at an angle \(\theta\) with the \(z\) axis.