Folding a Gaussian into Theoretical Data

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Abstract

We construct an approximation to a δ-function using a Gaussian of full width at half-maximum w. This approximate δ-function is used to model a measuring instrument. The Gaussian is folded into theoretical data and used to approximate the outcome expected when measuring the quantity described by the theory.

1 Method

Consider the function

$$\delta(x, w) = A \exp(-x^2/\Delta^2) \tag{1}$$

where

$$A = \frac{1}{w}\sqrt{\frac{\log(16)}{\pi}} \quad \& \quad \Delta = \frac{w}{\sqrt{\log(16)}}. \tag{2}$$

Note that $A\Delta = 1/\sqrt{\pi}$. The function $\delta(x, w)$, which serves as a finite-width approximation to the δ-function, satisfies

$$\int_{-\infty}^{\infty} \delta(x, w)\, dx = 1. \tag{3}$$

Fig. 1 shows the approximation function.

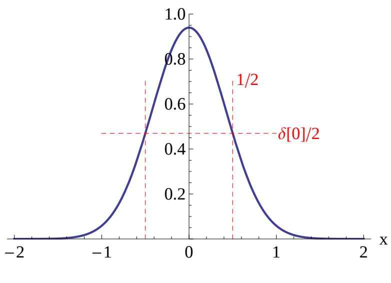

Figure 1: Finite-width approximation $\delta(x, w)$ to a δ-function. Case where $w = 1$.

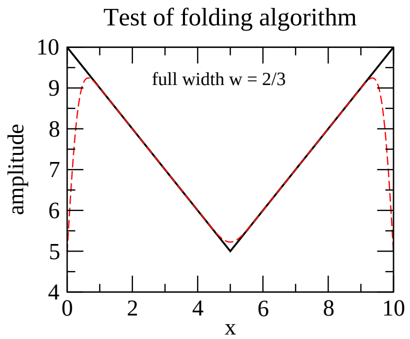

Figure 2: A Gaussian of width $w = 2/3$ is folded into a wedge function, which is shown in black, to give the result shown in red.

If we let $\delta(x, w)$ describe the the response of an instrument to a unit signal at value x, then the response $F(x, w)$ to a distributed signal with amplitude $f(x)$ will be

$$F(z, w) = \int_{-\infty}^{\infty} \delta(x, w)\, f(z - x)\, dx. \tag{4}$$

We approximate the above integral using using the trapezoidal rule. To make this rule tractable, we truncate the above integral at a point x_m where

$$\int_{-x_m}^{x_m} \delta(x, w)\, dx = 0.9999 \tag{5}$$

This point is $x_m = 1.6521822525771246w$. Thus, we approximate

$$F(z, w) \approx \int_{-x_m}^{x_m} \delta(x, w)\, f(z - x)\, dx. \tag{6}$$

We suppose that $f(x)$ is laid out on a grid $x_i = ih,\ i = 0 \cdot \cdot n.$ and that $\delta(x, w)$ is laid out on a grid $x_i = ih,\ i = 0 \cdot \cdot m$ The trapezoidal rule gives

$$F[x_i, w] = h \Bigg[\frac{1}{2} (f[x_i - x_m] + f[x_i + x_m])\, \delta[x_m, w] + f[x_i]\, \delta[0, w] + \sum_{j=1}^{m-1} (f[x_i - x_j] + f[x_i + x_j])\, \delta[x_j, w] \Bigg]. \tag{7}$$

We extend the array $f[x_i]$ from $i = 0 \cdot\cdot n$ to $i = -m \cdot\cdot n{+}m$ by adding subarrays of 0's before and after the original array. In this way, the transformed array $F[x_i, w]$ will be defined everywhere on the original grid. As an example, we show the result of folding a Gaussian into a wedge function in Fig. 2

In Fig. 3, we consider the bound-state contribution to the dynamic structure function for scattering at $\theta = 130°$ of a 9000 eV photon from a Ti metal

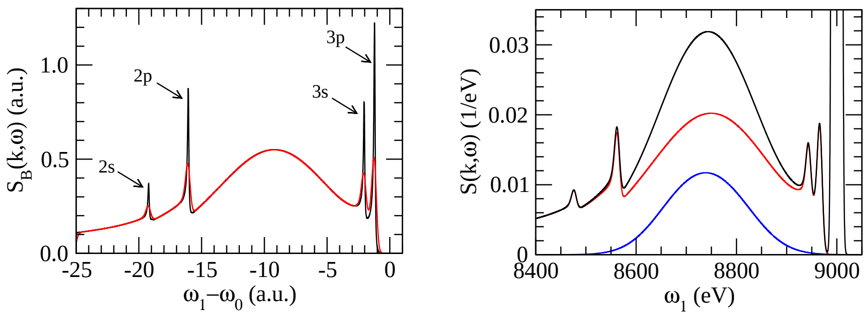

Figure 3: Left panel: (black) Theoretical bound-state contribution to the structure function for Thomson scattering at $\theta = 130°$ of a 9000 eV photon from Ti metal at $T = 10$ eV. (red) Theoretical structure function after folding in an instrumental width of 10 eV. Right panel: (blue) Free-free contribution, (red) bound-state contribution and (black) total structure function after a folding in a 10 eV instrument function.

plasma at $T = 10$ eV. The black curve in the left panel shows the theoretical structure function and the red curve shows the theoretical prediction after a 10eV instrumental width is folded in. The panel on the right shows the total structure function after folding in the Gaussian instrument function.