1 Distributed Magnetization

Let us assume that we have a nucleus with a distributed moment described by a magnetization vector $M(r)$ and magnetic moment $\mu$ related by

$$\mu = \int d^3 r M(r)$$

The vector potential of a point dipole with magnetic moment $\mu$:

$$A = \frac{\mu_0}{4\pi} \frac{\mu \times r}{r^3},$$

is then generalized to

$$A = \frac{\mu_0}{4\pi} \int d^3 s \frac{M(s) \times (r - s)}{|r - s|^3}.$$  

Let us suppose that $M(r)$ points along the $z$-axis and that its magnitude depends only on $r$. Then, $\mu = \mu \hat{z}$ with

$$\mu = 4\pi \int dr r^2 M(r).$$

We may rewrite the vector potential as

$$A = \frac{\mu_0}{4\pi} \hat{z} \times \int d^3 s M(s) \frac{r - s}{|r - s|^3}. \quad (1)$$

This can be conveniently rewritten as

$$A = -\frac{\mu_0}{4\pi} \hat{z} \times \nabla \Phi_M(r), \quad (2)$$

where the magnetic scalar potential is $\Phi_M(r)$ is defined by

$$\Phi_M(r) = \int d^3 s \frac{M(s)}{|r - s|} = 4\pi \left[ \frac{1}{r} \int_0^r ds s^2 M(s) + \int_r^\infty ds s M(s) \right]. \quad (3)$$

One easily shows that

$$-\nabla \Phi_M(r) = \frac{r}{r^3} 4\pi \int_0^r ds s^2 M(s).$$

It follows that we may write the vector potential for distributed magnetization in the form

$$A = \frac{\mu_0}{4\pi} \frac{\mu \times r}{r^3} f(r), \quad (4)$$

where

$$f(r) = \frac{4\pi}{\mu} \int_0^r ds s^2 M(s) = \int_0^r ds s^2 M(s) \div \int_0^\infty ds s^2 M(s).$$
1.1 Uniform Distribution

If $M(r)$ is constant inside a sphere of radius $R$ and vanishes outside, then

$$f(r) = \begin{cases} \frac{r^3}{R^3}, & r \leq R \\ 1, & r > R \end{cases}$$

(5)

From this, it follows that

$$A = \frac{\mu_0}{4\pi} \begin{cases} \frac{\mu \times r}{R^3}, & r \leq R \\ \frac{\mu \times r}{r^3}, & r > R. \end{cases}$$

(6)

A simple prescription to use in this case is to let

$$\frac{1}{r^2} \to \frac{r}{R^3}, \quad r < R$$

in the point dipole formula.

1.2 Fermi Distribution

Let $M(r)$ be described by a Fermi distribution:

$$M(r) = \frac{M_0}{1 + \exp((r - c)/a)}.$$  

(7)

The total magnetic moment is then given by

$$\mu = 4\pi \left[ \frac{e^3}{3} + \sum_{n=1}^{\infty} (-1)^n e^{-nc/a} \int_0^c ds s^2 e^{ns/a} \
- \sum_{n=1}^{\infty} (-1)^n e^{nc/a} \int_c^\infty ds s^2 e^{-ns/a} \right]$$

(8)

From Maple, we obtain

$$\int_0^c ds s^2 e^{ns/a} = e^{nc/a} \left( \frac{ac^2}{n} - 2a^2c \frac{a^2c}{n^2} + 2a^3 \frac{a^3}{n^3} \right) - 2a^3 \int_1^n \frac{(-1)^n}{n^3}$$

(9)

so that the first sum becomes

$$\sum_{n=1}^{\infty} (-1)^n e^{-nc/a} \int_0^c ds s^2 e^{ns/a} = ac^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$-2a^2c \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} + 2a^3 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} - 2a^3 \int_1^n \frac{(-1)^n}{n^3} e^{-nc/a}. \quad (10)$$

For the second integral, we obtain

$$\int_c^\infty ds s^2 e^{-ns/a} = e^{-nc/a} \left( \frac{ac^2}{n} + 2a^2c \frac{a^2c}{n^2} + 2a^3 \frac{a^3}{n^3} \right)$$

(11)
so the second sum becomes
\[
\sum_{n=1}^{\infty} (-1)^n e^{nc/a} \int_c^{\infty} ds \, s^2 e^{-ns/a} = ac^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} + 2a^2 e \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} + 2a^3 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}.
\]

Combining, we find
\[
\mu = 4\pi M_0 \left[ \frac{c^3}{3} - 4a^2 c \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - 2a^3 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} e^{-nc/a} \right].
\]

Making use of the fact that
\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12},
\]
and defining
\[
S_k(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^k} e^{-nx},
\]
we may rewrite the expression above as
\[
\mu = \frac{4\pi}{3} c^3 M_0 \left[ 1 + \frac{a^2}{c^2} \frac{\pi^2}{3} + 6 \frac{a^3}{c^3} S_3 \left( \frac{c}{a} \right) \right].
\]

Now, we may evaluate the factor \( f(r) \) in Eq. (5) using Maple
\[
f(r, r < c) = \frac{4\pi M_0}{\mu} \left[ \frac{r^3}{3} + \sum_{n=1}^{\infty} (-1)^n e^{-nc/a} \int_0^r ds \, s^2 e^{ns/a} \right]
\[
= \frac{4\pi M_0}{\mu} \left[ \frac{r^3}{3} - ar^2 S_1 \left( \frac{c-r}{a} \right) + 2a^2 r S_2 \left( \frac{c-r}{a} \right) - 2a^3 S_3 \left( \frac{c-r}{a} \right) \right].
\]

Similarly, again using Maple, we find
\[
f(r, r > c) = \frac{4\pi M_0}{\mu} \left[ \frac{c^3}{3} + \frac{a^2 c}{3} \frac{\pi^2}{3} + 2a^3 S_3 \left( \frac{c}{a} \right) \right.
\]
\[
- ar^2 S_1 \left( \frac{r-c}{a} \right) - 2a^2 r S_2 \left( \frac{r-c}{a} \right) - 2a^3 S_3 \left( \frac{r-c}{a} \right) \left. \right]
\]

These expressions may be simplified somewhat to give
\[
f(r, r < c) = \frac{1}{N} \left[ \frac{r^3}{c^3} - \frac{3}{3} \frac{ar^2}{c^3} S_1 \left( \frac{c-r}{a} \right) + 6 \frac{a^2 r}{c^3} S_2 \left( \frac{c-r}{a} \right) - 6 \frac{a^3}{c^3} S_3 \left( \frac{c-r}{a} \right) \right],
\]

\[
f(r, r < c) = \frac{1}{N} \left[ \frac{r^3}{c^3} - \frac{3}{3} \frac{ar^2}{c^3} S_1 \left( \frac{c-r}{a} \right) + 6 \frac{a^2 r}{c^3} S_2 \left( \frac{c-r}{a} \right) - 6 \frac{a^3}{c^3} S_3 \left( \frac{c-r}{a} \right) \right],
\]

\[
(14)
\]

\[
(15)
\]

\[
(16)
\]

\[
(17)
\]
Figure 1: Upper panel: \( f(r) \) and \( 50 f(r)/r^2 \) are shown for a Fermi distribution with \( c = 5.748 \) fm and \( t = 2.3 \) fm. Lower panel: \( g(r) \) and \( 500 g(r)/r^3 \) are shown for a Fermi distribution with \( c = 5.748 \) fm and \( t = 2.3 \) fm.

\[
\begin{align*}
f(r, r > c) &= 1 - 
\frac{1}{N} \left[ 3 \frac{a^2 r}{c^3} S_1 \left( \frac{r - c}{a} \right) + 6 \frac{a^2 r}{c^3} S_2 \left( \frac{r - c}{a} \right) + 6 \frac{a^3}{c^3} S_3 \left( \frac{r - c}{a} \right) \right], \quad (18)
\end{align*}
\]

where \( N \) is given by

\[
N = \left[ 1 + \frac{a^2}{c^2} \pi^2 + 6 \frac{a^3}{c^3} S_3 \left( \frac{c}{a} \right) \right].
\]

In the upper panel of Fig. 1, we plot the magnetic dipole scale factor \( f(r) \) and the function \( f(r)/r^2 \) occurring in hyperfine integrals.

## 2 Distributed Quadrupole Moment

Now let us suppose that the nuclear quadrupole moment is distributed over the nucleus according to some radial distribution function \( \rho(r) \). To analyze the
resulting potential, we first consider a point quadrupole. The point quadrupole potential is given by
\[ \Phi(r) = \frac{1}{2} \sum_{ij} \frac{Q_{ij}}{4\pi \epsilon_0} \frac{x_i x_j}{r^5}. \]

Since the trace of \( Q_{ij} \) vanishes, we may replace
\[ \frac{x_i x_j}{r^5} \rightarrow \frac{1}{3} \partial_j \partial_j \left( \frac{1}{r^3} \right) \]
in the expression for the potential. It follows that the potential of a quadrupole distributed symmetrically over the nucleus may be written
\[ \Phi(r) = \frac{1}{6} \sum_{ij} \frac{Q_{ij}}{4\pi \epsilon_0} \partial_j \partial_j \int \frac{4\pi x^2 \rho(x)}{|r - x|} dx, \]
where \( Q_{ij} \rho(r) \) is the distributed quadrupole moment density. The moment is normalized by requiring
\[ \int_0^\infty 4\pi x^2 \rho(x) = 1. \]

### 2.1 Uniform Distribution

Assuming that the distribution function \( \rho(r) = \rho_0 \) is constant over the nuclear volume, we have
\[ \rho_0 = \frac{3}{4\pi R^3} \]
where \( R \) is the nuclear radius, and
\[ \int \frac{4\pi x^2 \rho(x)}{|r - x|} dx = \begin{cases} \frac{1}{R} \left( \frac{3}{2} - \frac{r^2}{2R^2} \right), & r < R \\ \frac{1}{r}, & r > R \end{cases} \]
Differentiating and dropping terms proportional to \( \delta_{ij} \), we find
\[ \Phi = \frac{1}{2} \sum_{ij} \frac{Q_{ij}}{4\pi \epsilon_0} \frac{x_i x_j}{r^3} \ g(r), \]
where
\[ g(r) = \begin{cases} 0, & r < R \\ 1, & r > R \end{cases} \]

### 2.2 Fermi Distribution

For a spherically symmetric distribution \( \rho(x) \), we may write
\[ \int \frac{4\pi x^2 \rho(x)}{|r - x|} dx = 4\pi \left[ \frac{1}{r} \int_0^r x^2 \rho(x) dx + \int_r^\infty x^2 \rho(x) dx \right]. \]
Operating on this term with $\partial_i \partial_j$ leads to two terms, one proportional to $x_i x_j$ and one proportional to $\delta_{ij}$. Only the former term is of interest here. We pick out the coefficient of $x_i x_j$ using

$$\partial_i \partial_j F(r) \rightarrow x_i x_j \times \frac{1}{r} \frac{d}{dr} \frac{d}{dr} F(r).$$

It follows that

$$\frac{1}{3} \partial_i \partial_j \int \frac{4\pi x^2 \rho(x)}{|r - x|} dx \rightarrow \frac{4\pi x_i x_j}{r^3} \left[ \int_0^r x^2 \rho(x) dx - \frac{r^3}{3} \rho(r) \right].$$

The potential for the distributed moment can therefore be written

$$\Phi(r) = \frac{1}{2} \sum_{ij} \frac{Q_{ij}}{4\pi \rho_0} \frac{x_i x_j}{r^3} g(r).$$

with

$$g(r) = 4\pi \left[ \int_0^r x^2 \rho(x) dx - \frac{r^3}{3} \rho(r) \right]$$

The two screening functions $f(r)$ and $g(r)$ are seen to be identical, except for the second term in Eq. (19)!

Now, let us determine $g(r)$ for a Fermi distribution

$$\rho(r) = \frac{\rho_0}{1 + e^{(r-c)/\alpha}}.$$

Carrying out the integrations in Eq. (19), we find

$$g(r, r < c) = \frac{1}{N} \left[ \frac{r^3}{c^3} \frac{1}{1 + e^{(c-r)/\alpha}} + 6 \frac{a^3}{c^3} S_3 \left( \frac{c}{a} \right) - 6 \frac{a^3}{c^3} S_3 \left( \frac{c-r}{a} \right) \right.
+ 6 \frac{a^2 r}{c^3} S_2 \left( \frac{c-r}{a} \right) - 3 \frac{ar^2}{c^3} S_1 \left( \frac{c-r}{a} \right) \right],$$

and

$$g(r, r > c) = 1 - \frac{1}{N} \left[ \frac{r^3}{c^3} \frac{1}{1 + e^{(r-c)/\alpha}} + \frac{a^3}{c^3} S_3 \left( \frac{r-c}{a} \right) \right.
+ 6 \frac{a^2 r}{c^3} S_2 \left( \frac{r-c}{a} \right) + 3 \frac{ar^2}{c^3} S_1 \left( \frac{r-c}{a} \right) \right].$$

In the above formulas, the normalization constant $N$ is given by

$$N = \left[ 1 + \frac{a^2}{c^2} \pi^2 + 6 \frac{a^3}{c^3} S_3 \left( \frac{c}{a} \right) \right]$$

(22)
The functions $S_n(x)$, as before, are defined by

$$S_n(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^n} e^{-kx} \equiv -\text{Li}_n(-e^{-x}) \equiv -\text{Polylog}(n, -e^{-x})$$

It might be noted that

$$S_1(x) = \log(1 + e^{-x}).$$

For small $r$, one finds

$$g(r) \to \frac{e^{c/a} r^4}{12 a N (1 + e^{c/a})^2},$$

while for large $r$, $g(r) \to 1$. The function $g(r)$ is continuous at the point $r = c$. Indeed, the two forms are analytic continuations of a single function.

In the lower panel of Fig. 1, we plot the quadrupole scale factor $g(r)$ and the function $g(r)/r^3$ occurring in quadrupole integrals.

The functions $f(r)$ and $g(r)$ for a Fermi distribution are available numerically in the fortran subroutine nucfac.f.