Anomalous Moment Corrections to Transition Matrix Elements

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Introduction

It is shown on page 462 of Akhiezer and Berestetsky that radiative corrections induce the following modification of the current-field interaction energy:

$$\delta U_I(x) = \frac{e^3}{(4\pi)^2} \left(-\frac{1}{2mi}\right) \beta\, \gamma_\mu \gamma_\nu F_{\mu\nu}. \tag{1}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor. This can be rewritten in terms of electric and magnetic fields as:

$$\delta U_I(x) = -\frac{e^2}{4\pi} \frac{1}{2\pi} \frac{e}{2m} \left[\beta\, \vec{\Sigma} \cdot \vec{B} - i\beta\, \vec{\alpha} \cdot \vec{E}/c\right]. \tag{2}$$

For a static magnetic field $\vec{B}$, this reduces to

$$\delta U_I = -\delta\mu\ \beta\ \vec{\Sigma} \cdot \vec{B}, \tag{3}$$

with

$$\delta\mu = \frac{e^2}{4\pi} \frac{1}{2\pi} \frac{e}{2m}. \tag{4}$$

Multipole Expansion

Let us consider the interaction of the anomalous moment with the photon field. Describe the electromagnetic field by a multipole expansion:

$$\vec{A} = \hat{\epsilon}\, e^{ik\cdot r} \tag{5}$$

$$= 4\pi \sum_{JM\lambda} i^{J-\lambda} \left(\vec{Y}^{(\lambda)}_{JM}(\hat{k}) \cdot \hat{\epsilon}\right) \vec{a}^{(\lambda)}_{JM}(\hat{r}) \tag{6}$$

where

$$\begin{aligned}
\vec{a}^{(0)}_{JM}(\hat{r}) &= j_J(kr)\vec{Y}^{(0)}_{JM}(\hat{r}) && (7)\\
\vec{a}^{(1)}_{JM}(\hat{r}) &= \left[j'_J + \frac{j_J(kr)}{kr}\right]\vec{Y}^{(1)}_{JM}(\hat{r}) + \sqrt{J(J+1)}\,\frac{j_J(kr)}{kr}\vec{Y}^{(-1)}_{JM}(\hat{r})\ . && (8)
\end{aligned}$$

The above expression is in the transverse gauge. Since the fields are gauge independent, this gauge will lead to the same fields as any other. From page 217 of Varshalovich we find:

$$\begin{aligned}
\left[\vec{\nabla}\times\vec{a}^{(0)}_{JM}(\hat{r})\right] &= ik\ \vec{a}^{(1)}_{JM}(\hat{r}) && (9)\\
\left[\vec{\nabla}\times\vec{a}^{(1)}_{JM}(\hat{r})\right] &= -ik\ \vec{a}^{(0)}_{JM}(\hat{r}). && (10)
\end{aligned}$$

It follows that

$$\begin{aligned}
\vec{E} &= -\frac{\partial\vec{A}}{\partial t} = 4\pi\sum_{JM\lambda} i^{J-\lambda}\left(\vec{Y}^{(\lambda)}_{JM}(\hat{k})\cdot\hat{\epsilon}\right)\vec{e}^{(\lambda)}_{JM}(\hat{r}) && (11)\\
\vec{B} &= \left[\vec{\nabla}\times\vec{A}\right] = 4\pi\sum_{JM\lambda} i^{J-\lambda}\left(\vec{Y}^{(\lambda)}_{JM}(\hat{k})\cdot\hat{\epsilon}\right)\vec{b}^{(\lambda)}_{JM}(\hat{r}), && (12)
\end{aligned}$$

where

$$\begin{aligned}
\vec{e}^{(0)}_{JM}(\hat{r}) &= i\omega\,\vec{a}^{(0)}_{JM}(\hat{r}) && (13)\\
\vec{e}^{(1)}_{JM}(\hat{r}) &= i\omega\,\vec{a}^{(1)}_{JM}(\hat{r}) && (14)\\
\vec{b}^{(0)}_{JM}(\hat{r}) &= ik\,\vec{a}^{(1)}_{JM}(\hat{r}) && (15)\\
\vec{b}^{(1)}_{JM}(\hat{r}) &= -ik\,\vec{a}^{(0)}_{JM}(\hat{r})\ . && (16)
\end{aligned}$$

The electron-photon field interaction energy is

$$U_I(x) = -e\vec{\alpha}\cdot\vec{A} - \frac{e\hbar}{2mc}\frac{\alpha}{2\pi}\left[\beta\,\vec{\Sigma}\cdot\vec{B} - i\beta\,\vec{\alpha}\cdot\vec{E}/c\right] \qquad (17)$$

Note that $\hbar/(mc) = \alpha\, a_0$, so that in atomic units,

$$\vec{\alpha}\cdot\vec{A} \to \vec{\alpha}\cdot\vec{A} + \frac{\alpha^2}{4\pi}\left[\beta\,\vec{\Sigma}\cdot\vec{B} - i\beta\,\vec{\alpha}\cdot\vec{E}/c\right]. \qquad (18)$$

Using the multipole expansion, we obtain

$$\begin{aligned}
U_I(x) = -e4\pi\sum_{JM\lambda} i^{J-\lambda}\left(\vec{Y}^{(\lambda)}_{JM}(\hat{k})\cdot\hat{\epsilon}\right)\Big\{&\vec{\alpha}\cdot\vec{a}^{(\lambda)}_{JM}\\
&+\frac{\alpha^2}{4\pi}\left[\beta\,\vec{\Sigma}\cdot\vec{b}^{(\lambda)}_{JM} - i\beta\,\vec{\alpha}\cdot\vec{e}^{(\lambda)}_{JM}/c\right]\Big\} \qquad (19)
\end{aligned}$$

The matrix element of the term in curly braces is

$$\frac{1}{r^2}\left(\begin{array}{c} iP_{\kappa'}\,\Omega_{\kappa'm'} \\ Q_{\kappa'}\,\Omega_{-\kappa'm'}\end{array}\right)^\dagger \left\{\left(\begin{array}{cc} 0 & \vec\sigma\cdot\vec a^{(\lambda)}_{JM} \\ \vec\sigma\cdot\vec a^{(\lambda)}_{JM} & 0\end{array}\right)\right.$$
$$+\frac{\alpha^2}{4\pi}\left[\left(\begin{array}{cc} \vec\sigma\cdot\vec b^{(\lambda)}_{JM} & 0 \\ 0 & -\vec\sigma\cdot\vec b^{(\lambda)}_{JM}\end{array}\right)\right.$$
$$\left.\left.+\left(\begin{array}{cc} 0 & -i\vec\sigma\cdot\vec e^{(\lambda)}_{JM}/c \\ i\vec\sigma\cdot\vec e^{(\lambda)}_{JM}/c & 0\end{array}\right)\right]\right\}\left(\begin{array}{c} iP_{\kappa}\,\Omega_{\kappa m} \\ Q_{\kappa}\,\Omega_{-\kappa m}\end{array}\right) \tag{20}$$

Now, we use the fact that

$$\vec b^{(\lambda)}_{JM} = (-1)^\lambda\, ik\, \vec a^{(1-\lambda)}_{JM} \tag{21}$$
$$\vec e^{(\lambda)}_{JM}/c = ik\, \vec a^{(\lambda)}_{JM}, \tag{22}$$

to write the matrix element above as

$$\frac{i}{r^2}\left\{-P_{\kappa'}Q_\kappa\,\langle\kappa'm'|\vec\sigma\cdot\vec a^{(\lambda)}_{JM}|-\kappa m\rangle + Q_{\kappa'}P_\kappa\,\langle-\kappa'm'|\vec\sigma\cdot\vec a^{(\lambda)}_{JM}|\kappa m\rangle\right.$$
$$+\frac{k\alpha^2}{4\pi}\left[(-1)^\lambda P_{\kappa'}P_\kappa\,\langle\kappa'm'|\vec\sigma\cdot\vec a^{(1-\lambda)}_{JM}|\kappa m\rangle\right.$$
$$+(-1)^{1+\lambda}Q_{\kappa'}Q_\kappa\,\langle-\kappa'm'|\vec\sigma\cdot\vec a^{(1-\lambda)}_{JM}|-\kappa m\rangle$$
$$\left.\left.-P_{\kappa'}Q_\kappa\,\langle\kappa'm'|\vec\sigma\cdot\vec a^{(\lambda)}_{JM}|-\kappa m\rangle - Q_{\kappa'}P_\kappa\,\langle-\kappa'm'|\vec\sigma\cdot\vec a^{(\lambda)}_{JM}|\kappa m\rangle\right]\right\} \tag{23}$$

The electric-dipole matrix element is that with $J=\lambda=1$.

With the aid of the general formulas:

$$\langle-\kappa'm'|\vec\sigma\cdot\vec C^{(-1)}_{JM}|\kappa m\rangle = -\langle\kappa'm'|C^J_M|\kappa m\rangle \tag{24}$$
$$\langle-\kappa'm'|\vec\sigma\cdot\vec C^{(1)}_{JM}|\kappa m\rangle = \frac{\kappa-\kappa'}{\sqrt{J(J+1)}}\langle\kappa'm'|C^J_M|\kappa m\rangle \tag{25}$$
$$\langle\kappa'm'|\vec\sigma\cdot\vec C^{(0)}_{JM}|\kappa m\rangle = \frac{\kappa-\kappa'}{\sqrt{J(J+1)}}\langle\kappa'm'|C^J_M|\kappa m\rangle\ , \tag{26}$$

we find

$$\langle -\kappa' m' | \vec{\sigma} \cdot \vec{a}^{(1)}_{JM} | \kappa m \rangle = D \left\{ \frac{\kappa - \kappa'}{J+1} \left[j'_J + \frac{j_J}{kr} \right] - J \frac{j_J}{kr} \right\} \tag{27}$$

$$\langle \kappa' m' | \vec{\sigma} \cdot \vec{a}^{(1)}_{JM} | -\kappa m \rangle = D \left\{ -\frac{\kappa - \kappa'}{J+1} \left[j'_J + \frac{j_J}{kr} \right] - J \frac{j_J}{kr} \right\} \tag{28}$$

$$\langle \kappa' m' | \vec{\sigma} \cdot \vec{a}^{(0)}_{JM} | \kappa m \rangle = D \left\{ \frac{\kappa - \kappa'}{J+1} \, j_J(kr) \right\} \tag{29}$$

$$\langle -\kappa' m' | \vec{\sigma} \cdot \vec{a}^{(0)}_{JM} | -\kappa m \rangle = D \left\{ -\frac{\kappa - \kappa'}{J+1} \, j_J(kr) \right\}, \tag{30}$$

where

$$D = \sqrt{\frac{(2J+1)(J+1)}{4\pi J}} \, \langle \kappa' m' | C^J_M | \kappa m \rangle \tag{31}$$

Combining, we find

$$\begin{aligned} \langle \kappa' m' | \vec{\alpha} \cdot \vec{a}^{(1)}_{JM} + \cdots | \kappa m \rangle = \frac{i}{r^2} \sqrt{\frac{(2J+1)(J+1)}{4\pi J}} \, \langle \kappa' m' | C^J_M | \kappa m \rangle \times \\ \left\{ (P_{\kappa'} Q_\kappa + Q_{\kappa'} P_\kappa) \, \frac{\kappa - \kappa'}{J+1} \left[j'_J + \frac{j_J}{kr} \right] + (P_{\kappa'} Q_\kappa - Q_{\kappa'} P_\kappa) \, J \frac{j_J(kr)}{kr} \right. \\ + \frac{\alpha^2 k}{4\pi} \left[- (P_{\kappa'} P_\kappa + Q_{\kappa'} Q_\kappa) \, \frac{\kappa - \kappa'}{J+1} \, j_J(kr) + (P_{\kappa'} Q_\kappa - Q_{\kappa'} P_\kappa) \times \right. \\ \left. \left. \frac{\kappa - \kappa'}{J+1} \left[j'_J + \frac{j_J}{kr} \right] + (P_{\kappa'} Q_\kappa + Q_{\kappa'} P_\kappa) \, J \frac{j_J(kr)}{kr} \right] \right\}. \end{aligned} \tag{32}$$

Electric dipole: It follows that the modified dipole matrix element $\langle \kappa' || t^{(1)} || \kappa \rangle$ is

$$\begin{aligned} \langle \kappa' || t^{(1)} || \kappa \rangle = \langle \kappa' || C^{(1)} || \kappa \rangle \times \\ \int_0^\infty dr \left\{ (P_{\kappa'} Q_\kappa + Q_{\kappa'} P_\kappa) \, \frac{\kappa - \kappa'}{2} \left[j'_1 + \frac{j_1}{kr} \right] + (P_{\kappa'} Q_\kappa - Q_{\kappa'} P_\kappa) \, \frac{j_1(kr)}{kr} \right. \\ + \frac{\alpha^2 k}{4\pi} \left[- (P_{\kappa'} P_\kappa + Q_{\kappa'} Q_\kappa) \, \frac{\kappa - \kappa'}{2} \, j_1(kr) + (P_{\kappa'} Q_\kappa - Q_{\kappa'} P_\kappa) \times \right. \\ \left. \left. \frac{\kappa - \kappa'}{2} \left[j'_1 + \frac{j_1}{kr} \right] + (P_{\kappa'} Q_\kappa + Q_{\kappa'} P_\kappa) \, \frac{j_1(kr)}{kr} \right] \right\}. \end{aligned} \tag{33}$$

Let us examine the low-frequency limit of this expression. We find

$$\langle \kappa' || t^{(1)} || \kappa \rangle = \frac{k}{3} \, \langle \kappa' || q^{(1)} || \kappa \rangle,$$

where

$$\langle\kappa'||q^{(1)}||\kappa\rangle = \langle\kappa'||C^{(1)}||\kappa\rangle \int_0^\infty dr \Big\{ (P_{\kappa'}P_\kappa + Q_{\kappa'}Q_\kappa)\, r$$
$$+\frac{\alpha^2}{4\pi}\Big[\ \frac{(\kappa'-\kappa)}{2}\ (P_{\kappa'}P_\kappa + Q_{\kappa'}Q_\kappa)\,(kr) + \big(\kappa-\kappa'\big)\,(P_{\kappa'}Q_\kappa - Q_{\kappa'}P_\kappa)$$
$$+\,(P_{\kappa'}Q_\kappa + Q_{\kappa'}P_\kappa)\Big]\Big\}. \tag{34}$$

Magnetic dipole: Similarly, the magnetic dipole matrix element is given by

$$\langle\kappa'||t^{(0)}||\kappa\rangle = \langle-\kappa'||C^{(1)}||\kappa\rangle \times$$
$$\int_0^\infty dr \left\{ (P_{\kappa'}Q_\kappa + Q_{\kappa'}P_\kappa)\ \frac{\kappa+\kappa'}{2}\ j_1(kr)\right.$$
$$+\frac{\alpha^2\,k}{4\pi}\ \left[(P_{\kappa'}P_\kappa + Q_{\kappa'}Q_\kappa)\ \frac{\kappa+\kappa'}{2}\ \left[j_1'(kr) + \frac{j_1(kr)}{kr}\right]\right.$$
$$\left.\left.-\ (P_{\kappa'}P_\kappa - Q_{\kappa'}Q_\kappa)\,\frac{j_1(kr)}{kr} + \frac{\kappa+\kappa'}{2}\ (P_{\kappa'}Q_\kappa - Q_{\kappa'}P_\kappa)\ j_1(kr)\right]\right\}. \tag{35}$$

In the Pauli approximation, this reduces to

$$M \to \langle-\kappa'||C^{(1)}||\kappa\rangle\,\delta_{n'n}\,\delta_{l'l}\ \left[-\frac{(\kappa+\kappa')(\kappa+\kappa'-1)}{2} + \frac{\alpha}{2\pi}(\kappa+\kappa'-1)\right], \tag{36}$$

where we introduce the conventional magnetic-dipole matrix element M through the relation $\langle\kappa'||t^{(0)}||\kappa\rangle = \frac{\alpha}{2}\frac{k}{3}\,M$. As a specific example, we find

$$M = \frac{2}{\sqrt{3}}\left(1+\frac{\alpha}{\pi}\right)$$

for the fine-structure transition $np_{3/2} \to np_{1/2}$.

Appendix

Dirac Matrices

The basic Dirac α and β matrices are

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad \text{and} \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{37}$$

where σ_i are the 2×2 Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{38}$$

and where I is the 2×2 identity matrix. The Pauli matrices satisfy the multiplication rules

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k. \tag{39}$$

As a consequence,

$$\alpha_i \alpha_j = \delta_{ij} + i\epsilon_{ijk}\Sigma_k, \tag{40}$$

where the 4×4 sigma matrix is

$$\Sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}. \tag{41}$$

It follows that

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij}. \tag{42}$$

Furthermore,

$$\beta\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad \text{and} \qquad \alpha_i\beta = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}. \tag{43}$$

As a consequence

$$\alpha_i\beta + \beta\alpha_i = 0. \tag{44}$$

Now, we introduce the Dirac γ matrices as

$$\gamma_i = -i\beta\alpha_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix} \qquad \gamma_4 = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{45}$$

These matrices satisfy the relations

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}. \tag{46}$$

Dirac Equation

The time-dependent Dirac equation is written ($\hbar = 1$)

$$\left[\, c\,\vec{\alpha}\cdot\vec{p} + \beta mc^2 + V(r)\,\right]\psi = -\frac{1}{i}\frac{\partial\psi}{\partial t} \tag{47}$$

This can be recast in terms of the gamma matrices as

$$\left[c\,\vec{\gamma}\cdot\vec{\nabla} + mc^2 + \gamma_4 V\right]\psi = -c\,\gamma_4\frac{\partial\psi}{\partial x_4}. \tag{48}$$

Introducing $V = e\phi$ and $A_\mu^{\rm ext} = (\vec{A}, i\phi/c)$, we may rewrite this as

$$\gamma_\mu\left(\frac{\partial\psi}{\partial x_\mu} - ie\,A_\mu^{\rm ext}\,\psi\right) + mc\,\psi = 0. \tag{49}$$

Now, let's look at the adjoint equation:

$$\left(\frac{\partial\psi^\dagger}{\partial x_\mu^*} + i\psi^\dagger\, e\, A_\mu^{*\mathrm{ext}}\right)\gamma_\mu + mc\,\psi^\dagger = 0. \tag{50}$$

Introducing $\bar\psi = \psi^\dagger\gamma_4$, we find

$$\left(\frac{\partial\bar\psi}{\partial x_\mu} + i\bar\psi\, e\, A_\mu^{\mathrm{ext}}\right)\gamma_\mu - mc\,\bar\psi = 0. \tag{51}$$

Current and Interaction Hamiltonian

The electron four-vector current is ($e = -|e|$),

$$j_\mu = iec\,\bar\psi\gamma_\mu\psi. \tag{52}$$

Here

$$j_i = i\,ec\,\bar\psi\gamma_i\psi = ec\,\psi^\dagger\alpha_i\psi \tag{53}$$

$$j_4 = i\,ec\,\bar\psi\gamma_4\psi = iec\,\psi^\dagger\psi = ic\,\rho. \tag{54}$$

The Hamiltonian describing the interaction with an electromagnetic field described by A_μ is

$$\mathcal{H}_I = -j_\mu A_\mu = -\vec{j}\cdot\vec{A} + \rho\,\phi \tag{55}$$

It follows from the Dirac equation that current is conserved:

$$\frac{\partial j_\mu}{\partial x_\mu} = 0\,. \tag{56}$$

Maxwell Equations

We use SI units and write Maxwell's equations as

$$\vec\nabla\cdot\vec{E} = \frac{1}{\epsilon_0}\rho \tag{57}$$

$$\vec\nabla\cdot\vec{B} = 0 \tag{58}$$

$$\vec\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{59}$$

$$\vec\nabla\times\vec{B} = \mu_0\vec{j} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \tag{60}$$

Introduce potentials through

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{61}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}. \tag{62}$$

Require the Lorentz condition:

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0. \tag{63}$$

The inhomogeneous Maxwell equations become:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = -\frac{1}{\epsilon_0}\,\rho \tag{64}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0\,\vec{j}. \tag{65}$$

We can write the Lorentz condition in the form:

$$\frac{\partial A_\mu}{\partial x_\mu} = 0, \tag{66}$$

where

$$A_\mu = \left(\vec{A},\ i\frac{1}{c}\phi\right). \tag{67}$$

Introduce the field tensor

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \tag{68}$$

Explicitly,

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1/c \\ -B_3 & 0 & B_1 & -iE_2/c \\ B_2 & -B_1 & 0 & -iE_2/c \\ iE_1/c & iE_2/c & iE_3/c & 0 \end{pmatrix} \tag{69}$$

The inhomogeneous equations may be written

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0\, j_\mu \tag{70}$$