Note on Uehling Potential

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The vacuum-polarization correction for an electron in a nuclear Coulomb field can be described, to the neglect of terms of relative order $(\alpha Z)^2$, by a correction to the Coulomb potential

$$\delta V(r) = -\frac{2\alpha Z}{3\pi r} \int_1^\infty dt \sqrt{t^2-1} \left(\frac{1}{t^2} + \frac{1}{2t^4} \right) e^{-2ctr}. \tag{1}$$

This correction is referred to as the Uehling Potential [1]. The formula for the Uehling potential given in Eq. (1) was obtained from Eq. (44) of Wichmann and Kroll, Ref. [2], using the transformation $t = \sqrt{y^2+1}$.

For a charge distribution $\rho(r)$ normalized to

$$\int d^3x \, \rho(x) = 4\pi \int_0^\infty x^2 \rho(x) \, dx = Z, \tag{2}$$

the Uehling potential may be generalized to

$$\delta V(r) = -\frac{2\alpha}{3\pi} \int d^3x \, \rho(x) \int_1^\infty dt \sqrt{t^2-1} \left(\frac{1}{t^2} + \frac{1}{2t^4} \right) \frac{e^{-2ctR}}{R} \tag{3}$$

with $R = |\vec{r} - \vec{x}| = \sqrt{r^2 - 2rx\cos\theta + x^2}$. The integral over θ and ϕ, the polar angles of $\vec{x}$, can be expressed as

$$J(x,r) = \int_0^{2\pi} \int_0^{\pi} d\Omega \, \frac{e^{-\lambda R}}{R} = 2\pi \int_{-1}^{1} d\mu \frac{e^{-\lambda\sqrt{r^2-2rx\mu+x^2}}}{\sqrt{r^2-2rx\mu+x^2}}. \tag{4}$$

Using R as independent variable, we find

$$J(x,r) = -\frac{2\pi}{rx} \int_{r+x}^{|r-x|} dR \, e^{-\lambda R} = \frac{2\pi}{\lambda r x} \left[e^{-\lambda|r-x|} - e^{-\lambda(r+x)} \right]. \tag{5}$$

We may, therefore, express δV as

$$\delta V(r) = -\frac{2\alpha^2}{3r}\int_0^\infty dx\, x\, \rho(x) \int_1^\infty dt\sqrt{t^2-1}\left(\frac{1}{t^3}+\frac{1}{2t^5}\right)\times \left(e^{-2ct|r-x|}-e^{-2ct(r+x)}\right). \qquad (6)$$

This form of the Uehling potential is given by Fullerton and Rinker in Ref. [3]. With the transformation $u = 1/t$, we can write

$$\int_1^\infty dt\sqrt{t^2-1}\left(\frac{1}{t^3}+\frac{1}{2t^5}\right)F(t) = \int_0^1 du\,\sqrt{1-u^2}\left(1+\frac{1}{2}u^2\right)F(\frac{1}{u}). \qquad (7)$$

Therefore, finally,

$$\delta V(r) = -\frac{2\alpha^2}{3r}\int_0^\infty dx\, x\, \rho(x)\int_0^1 du\,\sqrt{1-u^2}\left(1+\frac{1}{2}u^2\right)\times \left(e^{-2c|r-x|/u}-e^{-2c(r+x)/u}\right). \qquad (8)$$

The integral over u may be carried out numerically using the adaptive Gaussian quadrature routine DQAGP from NETLIB. All of the above quantities are given in atomic units.

In our applications, we are particularly interested in cases where the function $\rho(r)$ is described by a Fermi-distribution

$$\rho(r) = \frac{\rho_0}{1+\exp\left[(r-c)/a\right]}, \qquad (9)$$

where c is the 50% radius of the distribution and where $a = t/[4\ln 3]$; t being the 10%–90% falloff interval. The charge distribution normalization factor is

$$\rho_0 = \frac{3Z}{4\pi c^3 \mathcal{N}}, \qquad (10)$$

where

$$\mathcal{N} = 1 + \pi^2\frac{a^2}{c^2} + 6\frac{a^3}{c^3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}e^{-na/c}. \qquad (11)$$

References

[1] E. A. Uehling, Phys. Rev. **48**, 55 (1935).

[2] E. H. Wichmann and N. H. Kroll, Phys. Rev. **101**, 843 (1956).

[3] L. W. Fullerton ang G. A. Rinker, Jr., Phys. Rev. A**13**, 1283 (1976).