Some examples of solved paradoxes

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1 The missing dollar paradox

One example of a paradox which you might have heard is the following (this is the Wikipedia version):

> "Three men go to a hotel room to stay. They receive a bill for \$30. They each put \$10 on the table, which the bellboy collects and takes to the till. The hotel manager informs the bellboy that the bill should only have been for \$25 and returns \$5 to the bellboy in \$1 bills. On the way back to the room the bellboy realizes that he cannot divide the bills equally between the men. As they didnt know the total of the revised bill, he dishonestly decides to put \$2 in his own pocket and give each of the men \$1. Now that each man has been given a dollar back, each of the men has paid \$9. Three times 9 is 27. The bellboy has \$2 in his pocket. Two plus 27 is \$29. The men originally handed over \$30. Where is the missing dollar?"

A first step toward seeing what is going on here is putting this story into the form of an argument. A first question is: what is the argument an argument for? What should its conclusion be? A plausible idea is that we can think of the above story as an argument that somewhere in these transactions a dollar has mysteriously vanished. Of course, this is absurd; but that's OK, since we want the conclusion of a good paradox to be apparently false.

What should the premises be? The idea here is to think of the various claims made in the story as premises. Here's an attempt to lay out the first part of the story:

1. At the start of the story — call this time t0 — each of the three men has paid \$10 to the bellboy.
2. The total amount of money the bellboy has at t0 is \$30. (1)

So far, so good. Now move on to the next part of the story:

3. At t1, the bellboy gives \$1 each back to each of the men and pockets \$2.
4. At t1, each of the men has spent \$9, for a total of \$27. (1,3)
5. At t1, the bellboy has \$2 in his pocket. (3)
6. At t1, the sum of what the men have spent and what is in the bellboy's pocket is \$29. (4,5)

Now let's see if we can put these two together to yield an argument for the conclusion that a dollar has vanished:

1. At the start of the story — call this time t0 — each of the three men has paid \$10 to the bellboy.
2. The total amount of money spent by the men at t0 is \$30. (1)
3. At t1, the bellboy gives \$1 each back to each of the men and pockets \$2.
4. At t1, each of the men has spent \$9, for a total of \$27. (1,3)
5. At t1, the bellboy has \$2 in his pocket. (3)
6. At t1, the sum of what the men have spent and what is in the bellboy's pocket is \$29. (3)
7. The money spent by the men at t0 = the sum of what the men have spent and what is in the bellboy's pocket at t1.

C. \$30 = \$29. (2,6,7)

The conclusion has got to be false; but if it is, we know that either a premise must be false or there must be a problem with the reasoning. Which is it?

2 Proof that 2=1

1.	$a, b \neq 0$	
2.	$a = b$	
3.	$a^2 = ab$	(2)
4.	$a^2 - b^2 = ab - b^2$	(3)
5.	$(a-b)(a+b) = b(a-b)$	(4)
6.	$(a+b) = b$	(5)
7.	$(b+b) = b$	(2,6)
8.	$2b = b$	(7)
9.	$(2b/b) = (b/b)$	(1,8)
C.	2=1	(9)

In this argument, we deduce an absurd conclusion from two apparently non-absurd assumptions. What has gone wrong? This is an example of a paradox which is solved by finding a flaw in the reasoning, rather than by rejecting a premise.

3 The barber

In the introduction to *Paradoxes*, Sainsbury tells the following story:

> "in a certain remote Sicilian village, approached by a long ascent up a precipitous mountain road, the barber shaves all and only those villagers who do not shave themselves. Who shaves the barber? If he himself does, then he does not (since he shaves *only* those who do not shave themselves); if he does not, then he indeed does (since he shaves *all* those who do not shave themselves)."

You can think of this as an argument for the absurd conclusion that there is a barber who both does and does not shave himself. How would you lay out the steps of that argument? Is the problem with a premise, or with the reasoning?

4 Galileo's paradox

In *Two New Sciences*, Galileo gave the following argument:

> "you also know that just as the products are called squares so the factors are called sides or roots; while on the other hand those numbers which do not consist of two equal factors are not squares. Therefore if I assert that all numbers, including both squares and non-squares, are more than the squares alone, I shall speak the truth, shall I not? ...
>
> If I should ask further how many squares there are one might reply truly that there are as many as the corresponding number of roots, since every square has its own root and every root its own square, while no square has more than one root and no root more than one square. ...
>
> But if I inquire how many roots there are, it cannot be denied that there are as many as the numbers because every number is the root of some square. This being granted, we must say that there are as many squares as there are numbers because they are just as numerous as their roots, and all the numbers are roots. Yet at the outset we said that there are many more numbers than squares, since the larger portion of them are not squares. Not only so, but the proportionate number of squares diminishes as we pass to larger numbers, Thus up to 100 we have 10 squares, that is, the squares constitute 1/10 part of all the numbers; up to 10000, we find only 1/100 part to be squares; and up to a million only 1/1000 part; on the other hand in an infinite number, if one could conceive of such a thing, he would be forced to admit that there are as many squares as there are numbers taken all together.
>
> So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite;

neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former ...”

Let’s use N as a name for the set of natural numbers (1, 2, 3, ...) and S as a name for the set of squares of natural numbers (1, 4, 9, ...). The passage above is paradoxical because it seems that we have a very plausible argument for the conclusion that N is both larger than, and the same size as, S. To state the argument, it will be useful to introduce one piece of terminology: ‘proper subset.’ We’ll say that one set X is a proper subset of another Y if and only if every member of X is a member of Y, but Y has some members that are not members of X.

1. For any two sets X and Y, if every member of X is a member of Y, and Y has some members that are not members of X, then Y has more members than X.
2. Every member of S is a member of N, but there are some members of N that are not members of S (since not all natural numbers have as their square root another natural number).
3. N has more members than S. (1,2)
4. For any two sets X and Y, if you can match every member of X to a different member of Y, and can match every member of Y to a different member of X, then X and Y have the same number of members.
5. You can match every member of S to a different member of N (its positive square root).
6. You can match every member of N to a different member of S (its square).
7. N and S have the same number of members. (4,5,6)

C. N has both more members than S, and the same number of members as S. (3,7)

What solution does Galileo suggest to this paradox? What do you think the solution is?