PHIL 20229: Paradoxes

Given the topic of this course, a natural place to begin is with the question: What is a paradox?

In *Paradoxes*, Sainsbury gives the following definition of a paradox:

> A paradox is “an apparently unacceptable conclusion derived by apparently acceptable reasoning from apparently acceptable premises.”

This description is fine as far as it goes, but it also raises some questions. What are premises and conclusions? What does it mean for a premise or conclusion to be acceptable or unacceptable? What does it mean for reasoning to be acceptable or unacceptable?

A good place to start here is by thinking about arguments as divisible into two parts, premises and conclusion. We're all familiar in an informal way with arguments. Consider, for example, the following argument:

> Notre Dame's football team will win more games this year than they did last year. After all, they won only eight games last year, and there's at least nine teams on the schedule for next year that they will definitely beat.

You can distinguish here what the speaker is arguing for - the conclusion of the argument - from what the speaker is using to support that conclusion - the premises.

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Here the conclusion seems to be:

ND’s football team will win more games this year than they did last year.

Whereas the premises seem to be:

ND’s football team won eight games last year.

ND’s football team will beat at least nine teams this year.

In thinking about arguments, even simple arguments like this one, it is useful to have a way of writing them out which makes their premises and conclusion explicit.

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1. ND’s football team won eight games last year.
2. ND’s football team will beat at least nine teams this year.

C. ND’s football team will win more games this year than they did last year. (1,2)

Here the premises are numbered, the horizontal line marks the move from premises to conclusion, and the numbers after the conclusion indicate the premises from which the conclusion is supposed to follow.

What does it mean to say that the conclusion **follows from** the premises?

What we mean is that if the premises are true, then the conclusion must be true; or, equivalently, it is impossible for the premises to be true and the conclusion false. In this sense, if the conclusion follows from the premises, then the truth of the premises guarantees the truth of the conclusion.

When the conclusion of an argument follows from its premises in this sense, we say that the argument is **valid**.

It is important to see that saying that an argument is valid does not involve saying that the premises are true; it just involves saying that if the premises are true, then the conclusion must be as well.

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Consider, for example, the following argument:

> I am the greatest basketball player in the world.
>
> ---
>
> I am a better basketball player than LeBron James.

It is valid; but, sadly, the premise is not true.

We can use this discussion of arguments to illuminate Sainsbury's definition of a paradox.

> The conclusion of an argument is apparently unacceptable if and only if it is apparently false.
>
> The premises of an argument are apparently acceptable if and only if they are apparently true.
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> The reasoning involved in an argument is apparently acceptable if and only if it is apparently valid.

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The reasoning involved in an argument is apparently acceptable if and only if it is apparently valid.

So: a paradox just is a certain kind of argument. It is an argument which has the following three features: (1) its premises appear to be true; (2) its conclusion appears to be false; and (3) it appears to be valid.

Why use “appears to” and “apparently” in describing paradoxes? Why not just say that a paradox is an argument whose premises really are true, whose conclusion really is false, and which really is valid?

Because it is impossible for there to be such an argument; it is impossible for there to be an argument which has true premises **and** a false conclusion **and** is such that it is impossible for its premises to be true and its conclusion false. A paradox is problematic because it seems to be an example of an argument of this sort, and so seems to be an actual example of something which we know to be impossible.

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For this reason, we know that at least one of the three appearances catalogued above must be misleading. **To solve a paradox is to figure out which one of these appearances is the misleading one.** (Of course, there might be more than one.)

Hence there are three ways to solve a paradox:

1. Find a false premise.

2. Show why the conclusion, which seems false, is really true.

3. Show why the reasoning employed is, contrary to appearances, invalid.

Now that we know how to solve a paradox, let’s go on to solve a few.

The missing dollar paradox

One paradox which you may have heard, and which definitely can be solved, is the "missing dollar paradox." Here's the Wikipedia version of the paradox:

> Three men go to a hotel room to stay. They receive a bill for \$30. They each put \$10 on the table, which the bellboy collects and takes to the till. The hotel manager informs the bellboy that the bill should only have been for \$25 and returns \$5 to the bellboy in \$1 bills. On the way back to the room the bellboy realizes that he cannot divide the bills equally between the men. As they didn't know the total of the revised bill, he dishonestly decides to put \$2 in his own pocket and give each of the men \$1. Now that each man has been given a dollar back, each of the men has paid \$9. Three times 9 is 27. The bellboy has \$2 in his pocket. Two plus 27 is \$29. The men originally handed over \$30. Where is the missing dollar?

A first step toward seeing what is going on here is putting this story into the form of an argument. A first question is: what is the argument an argument for? What should its conclusion be? A plausible idea is that we can think of the above story as an argument that somewhere in these transactions a dollar has mysteriously vanished. Of course, this is absurd; but that's OK, since we want the conclusion of a good paradox to be apparently false.

What should the premises be? The idea here is to think of the various claims made in the story as premises.

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Here is an attempt to lay out what happens in the first part of the story:

1. At the start of the story - call this time T1 - each of the three men has paid $10.
2. The total amount of money spent by the men at T1 is $30. (1)

Now let's move on to the next part of the story, when the bellboy goes to return the money to the men. Let's call the time at which this takes place T2.

3. At T2, the bellboy gives each of the men $1 and pockets $2.
4. At T2, each of the men has spent $9, for a total of $27 spent. (1,3)
5. At T2, the bellboy has $2 in his pocket. (3)
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So far, we don't have a paradox; after all, although all the premises seem true, and all the reasoning appears valid, we do not seem to have derived any apparently false conclusions.

But this is easily fixed. After all, out of all of the money the men spent at T1, every dollar must either still be spent, or be in the bellboy's pocket (there's nowhere else for the money to be). This can be stated as a premise:

7. The total amount of money spent by the men at T1 = the sum of what the man have spent and what is in the bellboy's pocket at T2.

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The conclusion of this argument certainly seems to be false. Hence, either one of the premises must be false, or the reasoning used must be invalid.

Let's look first at the reasoning. There are five logical steps in this argument. Could any of them be invalid?

If each of these steps is valid, then at least one of the premises must be false. But in thinking about this, it is important to see that there are two very different kinds of premises in this argument.

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If each of these steps is valid, then at least one of the premises must be false. But in thinking about this, it is important to see that there are two very different kinds of premises in this argument.

Some of the premises -- the derived premises -- are supposed to follow from other premises.

Other premises -- the independent premises -- are not supported by any further premises.

Now: if the logic used in the argument is valid, and there is at least one false premise, **there must be at least one false independent premise**. Can you see why?

So we know - given that the conclusion of our argument is false, and given that the reasoning used is valid - that one of premises 1, 3, and 7 must be false. Which one is it?

This is an example of a paradox which is solved by identifying a premise which, though it initially seemed plausible, turns out to be false.

But not every paradox can be solved in this way.

A good example of a paradox which cannot plausibly be solved by rejecting a premise is the following proof that 2=1.

A proof that 2=1

1.	$a, b \neq 0$	
2.	$a = b$	
3.	$a^2 = ab$	(2)
4.	$a^2 - b^2 = ab - b^2$	(3)
5.	$(a-b)(a+b) = b(a-b)$	(4)
6.	$(a+b) = b$	(5)
7.	$(b+b) = b$	(2,6)
8.	$2b = b$	(7)
9.	$(2b/b) = (b/b)$	(1,8)
C.	2=1	(9)

Here the only independent premises are 1 and 2; and there is surely nothing incoherent about introducing variables a, b and stipulating that a=b and that neither is equal to zero.

But we know that the conclusion is false. This means that there must be some flaw in the reasoning; after all, **whenever the conclusion of an argument is false, there is either a false independent premise or an invalid logical step.**

Where is the flaw in the reasoning?

Both the proof that 2=1 and the paradox of the missing dollar are pretty frivolous paradoxes, in the sense that we don't really learn anything of importance when we see how the paradox is to be solved. Fortunately for both of us, there are other paradoxes with more important consequences.

Consider the following passage from Galileo's *Two New Sciences*, in which he is discussing the mathematics of the infinite:

> You also know that just as the products are called squares so the factors are called sides or roots; while on the other hand those numbers which do not consist of two equal factors are not squares. Therefore if I assert that all numbers, including both squares and non-squares, are more than the squares alone, I shall speak the truth, shall I not? ...
>
> If I should ask further how many squares there are one might reply truly that there are as many as the corresponding number of roots, since every square has its own root and every root its own square, while no square has more than one root and no root more than one square. ...
>
> But if I inquire how many roots there are, it cannot be denied that there are as many as the numbers because every number is the root of some square. This being granted, we must say that there are as many squares as there are numbers because they are just as numerous as their roots, and all the numbers are roots. Yet at the outset we said that there are many more numbers than squares, since the larger portion of them are not squares. Not only so, but the proportionate number of squares diminishes as we pass to larger numbers, Thus up to 100 we have 10 squares, that is, the squares constitute 1/10 part of all the numbers; up to 10000, we find only 1/100 part to be squares; and up to a million only 1/1000 part; on the other hand in an infinite number, if one could conceive of such a thing, he would be forced to admit that there are as many squares as there are numbers taken all together.
>
> So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all the numbers, nor the latter greater than the former ...

Here Galileo is considering two different sets of numbers. First, he is considering the set *N* of natural numbers:

N: 1, 2, 3, 4, 5, 6, 7, ...

Next, he is considering the set *S* of squares, i.e. the set of natural numbers whose positive square root is another natural number:

S: 1, 4, 9, 16, 25, 36, 49 ...

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It certainly seems that *N* is larger than *S*. After all, intuitively speaking, everything in *S* is also in *N*, but very few of the numbers in *N* are also in *S*. But Galileo gives an argument that, surprisingly, this is not true.

To state this argument, it will be useful to introduce two bits of terminology. Let's say that one set A is a **proper subset** of another set B when everything in A is also in B, but there are some things in B which are not in A. And let's say that there is a **one-to-one pairing** of sets A and B when there is some way of matching members of B with members of A which is such that every member of A gets matched with exactly one member of B, and every member of B gets matched with exactly one member of A.

Galileo's paradox can then be given as the following argument:

1. There is a one-to-one pairing of *S* and *N.*
2. If there is a one-to-one pairing of two sets, then the two sets have the same number of members.
3. *S* and *N* have the same number of members. (1,2)
4. *S* is a proper subset of *N*.

C. Some sets have the same number of members as sets they are proper subsets of. (3,4)

Galileo’s paradox

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This counts as a paradox because the premises all seem true, the reasoning appears to be valid, and yet the conclusion certainly seems to be false; given our definition of “proper subset,” it is hard to see how one set could be a proper subset of another **and** the two sets be the same size.

Despite this, Galileo’s view (which is now the accepted view) is that the conclusion of this argument is, contrary to appearances, true. This is thus an example of a paradox which is to be solved by accepting the conclusion of the argument. It is also, unlike our two previous examples, a paradox from which the lesson to be learned is a nontrivial one.

Next class, we will begin our discussion of the paradoxes of space and time. The paradoxes we’ll discuss are some of the first paradoxes ever discussed as such: Zeno’s paradoxes of motion.