Russell's reduction of mathematics to logic

Jeff Speaks

October 21, 2004

We have seen, though our discussion of the theory of descriptions, the sense datum theory of perception, and Russell's distinction between knowledge by acquaintance and knowledge by description, the beginnings of a comprehensive view of the relations between mind, language, and the world. Roughly, the view is that perception acquaints us with sense data, universals, and relations; our thoughts consist of combinations of these things; and the sentences which we understand have as their meanings thoughts of this kind. Sentences which we understand which seem not to fit this mold – such as sentences containing proper names for material objects – are analyzed according to the theory of descriptions as having the meanings of sentences which make mention of descriptions involving universals and relations rather than names of material objects. In this way, the theory of descriptions, and the distinction between surface form and logical form, comes to the aid of Russell's kind of empiricism.

However, this account of the nature of our thoughts and knowledge concerning the external world does not account for one important kind of knowledge: our knowledge of the truth of mathematics. The truths of mathematics can seem to be in need of explanation for at least two reasons:

- *Epistemology.* Even if experience can provide justification for our beliefs about the external world, this does not help us to explain why our beliefs about mathematics are justified. Such justification, after all, seems to be a priori. How is this possible?

- *Metaphysics*. The truths of mathematics seem to require the existence of a great variety of mathematical objects. But it is reasonable to wonder whether all the different kinds of numbers really exist. But how can we account for the truth of claims like '2 + 2 = 4' without positing the existence of such mysterious mathematical objects?

Russell's answer to these problems was that, just as claims about material objects are disguised claims about sense data and properties, so claims about mathematics are disguised claims about logic. This helps with the epistemological problem, since presumably our knowledge of logical truths is unproblematic. It helps with the metaphysical problem, since it allows us to avoid positing the existence of any distinctively mathematical objects.

Our aim will be to understand and evaluate Russell's claim that truths of mathematics are disguised truths of logic. In your reading, you should utilize not only the assigned Chapter 2 of Russell's *Introduction to Mathematical Philosophy*, but also the chapter from Scott Soames's history of analytic philosophy which is in the coursepack. We will be covering some material which Soames discusses, but which goes beyond the material in Russell's Chapter 2. (I will cite some of the other relevant primary sources as we go.)

1 Russell's definition of number

A first step in seeing how Russell aimed to show that mathematical truths are disguised versions of logical ones is to see what he thought numbers to be.

Russell thinks that the first step is getting clear on the 'grammar' of numbers:

> "Many philosophers, when attempting to define number, are really setting to work to define plurality, which is quite a different thing. *Number* is what is characteristic of numbers, as *man* is what is characteristic of men. A plurality is not an instance of number, but of some particular number. A trio of men, for example, is an instance of the number 3, and the number 3 is an instance of number; but the trio is not an instance of number. This point may seem elementary and scarcely worth mentioning; yet it has proved too subtle for the philosophers, with few exceptions."(11)

The basic idea here is that we need to distinguish between giving definitions of particular numbers, like the number 3, and giving a definition of number itself. Russell's basic idea here is that both the number 3 and number are 'characteristics', which you can think of as being kind of like properties. The number 3 is a property of a trio of men; number-hood is a property of the number 3. But these two properties are properties of different kinds of things. Russell is saying that the number 3 is a property of trios, whereas number is a property of the number 2, the number 3, and the other numbers. (12)

(Compare: redness, and color. My sweater is red, and redness is a color, but my sweater is not a color.)

It is at this stage in the discussion that Russell introduces the topic of *collections*, or *sets*. It is important to see what role sets are playing here. For many purposes, we can replace talk of properties, or characteristics, of things with talk of the set of things with that characteristic. For example, for some purposes we can replace the property of redness with the set of red things. This is sort of like what Russell is doing here. He began by talking about the number 3, and number in general, as properties or characteristics. Now he is moving from this to talking about the number 3, and number in general, as *sets*.

The next question is, what sets are they? Russell says:

> "Reurning now to the definition of number, it is clear that number is a way of bringing together certain collections, namely, those that have a given number of terms. We can suppose all couples in one bundle, all trios in another, and so on. In this way we obtain various bundles of collections, each bundle consisting of all the collections that have a certain number of terms. Each bundle is a class whose members are collections, *i.e.* classes; thus each is a class of classes." (14)

Thus the number two is the set of all sets with two members, the number three is the set of all three-membered sets, and so on.

How this fits the 'grammar' of the inquiry. 'Being an instance of the number 3' and 'being a member of the set which is the number 3.'

This is not a successful definition of the numbers, however; it uses the notion of a 'set with two members'. But 'two' is just what we are trying to define. So at this stage we know what kind of things Russell takes numbers to be – they are sets of sets – but we do not yet have a definition of the numbers.

A first step in doing this is to say when two sets have the same number of members. Russell says,

> "In actual fact, it is simpler logically to find out whether two collections have the same number of terms than it is to define what that number is. An illustration will make this clear. If there were no polygamy or polyandry anywhere in the world, it is clear that the number of husbands living at any moment would be exactly the same as the number of wives. We do not need a census to assure us of this, nor do we need to know what is the actual numbers of husbands and wives. We know that the number must be the same in both collections, because each husband has one wife and each wife has one husband. The relation of husband and wife is what is called "one-one."" (15)

As Russell goes on to point out, we can define the relation of 'having the same number of members' in terms of a one-one relation:

> S and S' have the same number of members $\equiv_{df}$ there is some one-one relation between S and S'

Russell says that when two sets have a one-one relation between them, they are *similar*.

Then, given any set S, we can define the *number of that class*as follows:

> N is the number of a class $S \equiv_{df} N$ is the set of all sets similar to S.

or, as Russell puts it,

"*The number of a class is the class of all those classes that are similar to it.*" (18)

This allows us to give a definition of what it is for something to be a number, which Russell expresses as follows:

"*A number is anything which is the number of some class.*" (18)

As Russell notes, this sounds circular, but isn't. To see that it is not circular, note that we can express it as follows:

N is a number $\equiv_{df}$ there is some set S such that N is the set of all sets similar to S.

N is a number $\equiv_{df}$ $\exists\, S\ \forall x$ (x is a member of $N \equiv x$ is similar to S)

This gives us a noncircular definition of number.

There is a sense in which we still have not given a noncircular definition of specific numbers, like the number 2. That will come a bit later, when we present the details of Russell's logical system, and the axioms of arithmetic. For now, the important thing is that you get the basic idea behind what kinds of things Russell thinks numbers are. Now we'll move on to try to say why he thinks that numbers are these kinds of things.

2 The idea of reducing one theory to another

The basic claim of *logicism* is that mathematics is really a branch of logic. This is sometimes expressed by saying that mathematics (in this case, arithmetic) is *reducible* to logic.

What must someone show in order to show that one theory is, in this sense, reducible to another?

2.1 Axioms and theories

To make this clear, we'll first have to get clearer on what kinds of things theories are.

We will take 'theories' to bet sets of sentences, which include a set of basic *axioms*. The axioms are expressed in terms of concepts which are undefined, and basic (so far as the theory is concerned).

Other sentences will follow from the axioms of the theory. These are called *theorems* of the theory.

There may also be another class of sentences, *definitions*, which define new expressions in terms of the vocabulary in the axioms. If a theory contains definitions as well as axioms, the theorems of the theory will include all sentences which follow from the axioms, together with the definitions.

2.2 Bridging definitions

Suppose that we have two theories, T_1 and T_2, each of which include a set of axioms. Suppose further, as is usually the case, that the axioms contain different privileged vocabulary. What must be the case for us to be able to say that T_2 is *reducible* to T_1?

A natural thought is that T_2 is reducible to T_1 just in case every theorem of T_2 follows from the axioms of T_1. This would mean, in effect, that everything which T_2 says is provable from the resources of T_2. Since all the theorems of a theory follow from its axioms, to show that T_2 is reducible to T_1 in this sense, all that is required is to show that the *axioms* of T_2 follow from the axioms of T_1.

But at this point, we reach a problem. Typically, the axioms of the theory to be reduced are given in different vocabulary from the axioms of the reducing theory. But then there is no obvious way to prove the axioms of one from the axioms of the other. The case we are interested in — the reduction of arithmetic to logic — is a case in point. As we will see, the axioms of arithmetic say things about numbers, such as zero, and relations between numbers, such as the relation of one number being the successor of another (i.e., the number after another). But the axioms of logical theories say nothing at all about such things. So how could we prove the axioms of arithmetic from the axioms of logic?

The answer is that we will need bridging definitions, which translate the vocabulary of the theory to be reduced into the vocabulary of the reducing theory. We can now see that Russell's definition of numbers in terms of sets might play this role. If the axioms of some logical theory mention sets, and sets are numbers, then maybe we can prove arithmetical truths from logical axioms after all.

One important question here is: how can we tell whether a bridging definition is a good one? This is a question to which we will return.

3 Russell's reduction of arithmetic to logic

We have have some grasp of Russell's definitions of the numbers, and have some grasp of what it means for one theory to be reducible to another. We want to get into position to be able to evaluate the claim that arithmetic is reducible to logic. To do this, we will have to get clearer on what the 'theories' of arithmetic and logic are like. And to do this, we will have to say what the axioms of arithmetic and logic are.

3.1 The axiomatization of arithmetic

3.2 Peano's five axioms

In Chapter 1, Russell gives the following statement of Peano's axioms of arithmetic:

> "(1) 0 is a number.
> (2) The successor of any number is a number.
> (3) No two numbers have the same successor.
> (4) 0 is not the successor of any number.

(5) Any property which belongs to 0, and also to the successor of every number which has the property, belongs to all numbers." (5-6)

We will follow Russell in taking these as the five axioms of arithmetic.

3.3 Addition

To get a handle on these axioms, it may be useful to see how simple arithmetic operations can be explained using them. We will use the example of addition. The question is: how can facts about addition be proved using these five axioms?

Addition can be defined using the following two claims (for any $x, y \in N$):

$$x + 0 = x$$
$$x + y' = (x + y)'$$

How can these two claims be used to show that, e.g., 1+2=3? How does this proof use axioms (1) and (2) above?

Why, intuitively, are axioms (3)-(5) needed?

To show that arithmetic is reducible to logic, we will have to show that these five axioms are provable on the basis of logical truths. So our next step is to examine the logical system that Russell is working with.

3.4 Russell's logical system

We will follow Soames (p. 140 ff.) in presenting first a simplified version of Russell's system, and then adding complications in response to problems with it.

Russell's logical system extends usual systems of logic by adding the primitive symbol '∈', which intuitively means 'is an element of' or 'is a member of.' Russell's logical system contains three axioms which include this symbol:

1. *The axiom schema of comprehension.*

   $\exists y \forall x \ (\mathrm{F}x \equiv x \in y)$

   Informally, this says that for every predicate in the language, there is a set of things which satisfy that predicate. As Soames puts it (141), "To think of this as a logical principle is, in effect, to think that talk about an individual x's being so and so is interchangeable with talk about x's being in the set of things which are so and so."

2. *The axiom of extensionality.*

   $\forall a \forall b \ [\forall x \ (x \in a \equiv x \in b) \rightarrow a = b]$

   If a and b are sets with the same members, they are the same set.

3. *The axiom of infinity.*

   $\emptyset \notin N$

   This axiom is meant to guarantee that an infinite number of objects exist; what it says is that the empty set is not a member of the set of natural numbers. Soon we will see why Russell needs an axiom of infinity, and why he expresses it in this counter-intuitive way.

It is worth noting, by the way, that the null set is *not* a new primitive in Russell's system. It is definable as the set which is such that everything is not a member of it. ($\emptyset$ = the S: $\forall x$ $(x \notin S)$)

3.5 Definitions of arithmetical terms

As noted earlier in our discussion of reductions of one theory to another, what we want is a proof of the axioms of arithmetic on the basis of the axioms of Russell's logical system. But, since the two have different primitive vocabularies, we will need a translation of the vocabulary of one into the vocabulary of the other. We will now see how the kind of definition of the numbers described by Russell in the chapter we read can be adapted to do this.

The main arithmetical concepts involved in the Peano axioms are *zero* , *successor*, and *natural number*. These can be defined as follows:

Definition of zero

$0 = \{\emptyset\}$

Zero is the set whose only member is the empty set.

Definition of successor

The successor of a set a is that set which contains every set which contains a member x such that, if x is eliminated from that set, what is left is a set which is a member of a.

So the successor of zero is the set which contains every set with the following characteristic: it contains a member such that, when that member is eliminated from the set, the remaining set is a member of zero. But there is only one member of zero: the empty set. So the successor of zero is a set which contains all sets which contains a member which, when eliminated from the set, yields the empty set. But this just is the set of all one-membered sets; one-membered sets are the ones which have a member which, when eliminated, yields the empty set. So the successor of zero – i.e., the number 1 – is the set of all one-membered sets. The number 2 is defined as the successor of 1, and so on.

It is important to note that this definition is not circular. All that we used were the definitions of zero and successor, and neither of these presuppose arithmetic concepts. To see this, note first the following three notions from set theory:

> *Complement.* The complement of a set S is the set of all things which are not members of S. 'The complement of S' is written 'Comp(S).'
>
> *Union.* The union of two sets is the set of all things in either set. 'The union of S and T' is written '$S \cup T$.'

Intersection. The intersection of two sets is the set of all things which are members of both sets. 'The intersection of S and T' is written '$S \cap T$.'

Given these, we can give the following definition of the successor of 0 (i.e., 1):

0' = the set of all sets S which meet the following condition: $\exists x$ ($x \in S$ & $[S \cap \text{Comp}(\{x\}) \in 0]$)

i.e., given our definition of zero as the set whose only member is the empty set,

0' = the set of all sets S which meet the following condition: $\exists x$ ($x \in S$ & $[S \cap \text{Comp}(\{x\}) \in \{\emptyset\}]$)

(The 'trick' here is that the set obtained by removing a term x from a set S is the intersection of S with the complement of the set whose only member is x (i.e, $\{x\}$.)

Definition of natural number

N = the smallest set containing zero and closed under successor.

3.6 A proof of the axioms of arithmetic in Russell's logical system

So far, we have laid out the axioms of Russell's system, and have shown how, within that system, we can give definitions of the primitive terms used in the axiomatization of arithmetic. We are now in a position to test those definitions by trying to prove the five axioms of arithmetic on the basis of Russell's three logical axioms, plus the definitions of zero, successor, and natural number.

(A more in depth version of these proofs is provided in Soames, pp. 146 ff.)

Proof that zero is a natural number

Given the definition of natural number above, this is trivially true.

Proof that the successor any number is a number

This is also trivial, given the definitional claim that the set of natural numbers is closed under successor.

Proof that zero is not the successor of any number

We can prove this by reductio ad absurdum. (To prove some proposition p by reductio ad absurdum is to show that p is true by reducing the negation of p to absurdity — i.e., to show that $\neg p$ implies a contradiction.)

1. Suppose that for some x, $x'=0$.
2. Then, by the definition of 0, for some x, $x' = \{\emptyset\}$.
3. Then, by the definition of successor, every member m of $\{\emptyset\}$ is such that it has a member such that, when removed from m, the remaining set is a member of x.
4. So every member of $\{\emptyset\}$ has a member.
5. So $\emptyset$ has a member.
6. But $\emptyset$ is the set with no members.

Proof that no two different numbers have the same successor

To prove this, we show that whenever x, y have the same successor, $x = y$. This can be proved in two stages.

1. First, suppose that the successor x, y is not the empty set. Then it – call it S – is a set with members. The definition of successor tells us that if S is the successor of x, then if we take any member m of S and eliminate one of its members, the resultant set m' is a member of x. Parallel reasoning shows that m' is a member of y. But we know from the definition of the numbers that if x, are numbers and have a member in common, then $x = y$.

2. How about the other possibility – that the successor of x, y is the empty set? To see why this is worth considering, it is important to see what would have to be true for the successor of some number to be the empty set. Recall that, e.g., the number 23 is the set of all 23-membered sets. (This is not the definition of the number – which is given in terms of successor – but it is equivalent to what the definition says.) Now suppose that there were only 23 objects in the universe. What would the successor of 23 be? The set of all 24-membered sets. But, if there are only 23 objects in the universe, there are no 24-membered sets. That means that the successor of 23, i.e. 24, would be the empty set. Then what would the success of 24 be? The set of all 25-membered sets. But, again, in the scenario being considered, there are none. So 25 would be the empty set, just like 24. So in this scenario, $23' = 24'$, even though $23 \neq 24$. But this conflicts with the axiom of arithmetic which we are trying to prove. Hence this scenario is one which must be ruled out by some axiom of Russell's logical system. And it is: by axiom 3, the Axiom of Infinity. This axiom says that the empty set is not a member of the set of natural numbers. Now you can see why this strange-sounding axiom is called the Axiom of Infinity: it in effect guarantees that there are infinitely many objects in the universe.

Proof of the validity of mathematical induction

(We will not be discussing the proof of this in class. A brief discussion may be found in Soames, pp. 147-148.)

4 The axiom of infinity, pt. 1

We have shown that the axioms of arithmetic are provable in Russell's logical system; given what we said above about the relations between the axioms and theorems of a theory, this suffices to show that the theorems of arithmetic are also provable in this system.

But you might still have the following doubt about the reduction: is Russell's system of logic really just a system of logic? This doubt is connected with the Axiom of Infinity. Perhaps it is true that there are infinitely many objects; but it does not seem to be a truth of logic that there are. How could we know on the basis of knowledge of logic alone how many things there are?

A response to this problem: iterated sets.

5 Russell's paradox and the axiom schema of comprehension

But there is a further problem, which Russell was the first to discover, with another of the axioms of the logical systems presented above. This problem is known as *Russell's paradox.*

Consider axiom 1, the Axiom Schema of Comprehension. Intuitively, this says that for any way which an object can be (i.e., for any property which an object can have), there is a set of those objects which are that way (i.e., there is a set of just those objects with that property).

Consider first the property of being a member of oneself. Intuitively, this is a property which some sets have, and some sets do not have. The set of all abstract objects, for example, is a member of itself, since sets are abstract objects and the set of all abstract objects is a set. But the set of red things is not a member of itself, since it is not red. So far, so good; there is this property of being a member of oneself which some sets have and some do not; so presumably there is a set of all those sets which are a member of themselves.

Now consider the opposite property: the property of not being a member of oneself. Since this too appears to be a way that some sets are and some are not, we should be able to talk about the set of all sets which are not a member of themselves. Call this set S:

$S =$ the set of all sets which are not members of themselves.

But now consider the question: is S a member of itself? Suppose first that it is. If it is, then S must not be a member of itself, since this is what it takes to be a member of S. But this is a contradiction. So it must be the case that S is not a member of itself. But then it must be a member of itself, since everything which is not a mmber of itself is in S. But this too is a contradiction. So S cannot exist. But this is shows that the unrestricted axiom schema of comprehension is false.

We can give the same argument more formally as follows.

1. $\exists y \forall x \; ((x \in y) \equiv (x \notin x))$
2. Let 'S' be a name for the set whose existence is stated by (1).
   Then $\forall x \; ((x \in \mathrm{S}) \equiv (x \notin x))$
3. $(S \in \mathrm{S}) \equiv (S \notin S)$

Since (3) follows from (1), and (3) is a contradiction, (1) is false.

Russell's solution to this problem: the theory of logical types.

6 The axiom of infinity, pt. 2

The problem that the theory of logical types poses for our understanding of the Axiom of Infinity. Why it shows that the Axiom of Infinity requires the existence of infinitely many objects which are not sets.

7 The paradox of analysis

The epistemological and metaphysical aims of the logicist reduction as dependent on the claim to have captured the meaning of arithmetical statements. A problem with this view; the paradox of analysis.

Analyses as opposed to 'explications,' which attempt to replace one concept with another, less problematic one which can do the same theoretical work. Some evidence that Russell, at the time of writing *Introduction to Mathematical Philosophy*, thought of his definitions of the numbers as explications rather than straightforward analyses:

> "But when we come to the actual definition of numbers we cannot avoid what must at first sight seem a paradox, though this impression will soon wear off. We naturally think that the class of couples (for example) is something different from the number 2. But there is no doubt about the class of couples: it is indubitable and not difficult to define, whereas the number 2, in any other sense, is a metaphysical entity about which we can never feel sure that it exists or that we have tracked it down. It is therefore more prudent to content ourselves with the class of couples, which we are sure of, than to hunt for a problematical number 2 which must always remain elusive.
>
> ... At the expense of a little oddity, this definition secures definiteness and indubitableness; and it is not difficult to prove that numbers so defined have all the properties that we expect numbers to have." (18)