Finance 30210  
Solutions to Problem Set #5: Consumer Demand Analysis

1) For each of the following demand curves, calculate the price elasticity of demand and the income elasticity of demand.

a) \( Q = 800 - 4P + 2I \)

Price Elasticity: \( \epsilon_p = \frac{\% \Delta Q}{\% \Delta P} = \left( \frac{\Delta Q}{\Delta P} \right) \left( \frac{P}{Q} \right) = -4 \left( \frac{P}{Q} \right) \)

Income Elasticity: \( \epsilon_i = \frac{\% \Delta Q}{\% \Delta I} = \left( \frac{\Delta Q}{\Delta I} \right) \left( \frac{I}{Q} \right) = 2 \left( \frac{I}{Q} \right) \)

b) \( \ln Q = 2.5 - .65 \ln P + .95I \)

Price Elasticity: \( \epsilon_p = \frac{\% \Delta Q}{\% \Delta P} = \left( \frac{\Delta \ln Q}{\Delta \ln P} \right) = -.65 \)

Income Elasticity: \( \epsilon_i = \frac{\% \Delta Q}{\% \Delta I} = \left( \frac{\Delta \ln Q}{\Delta I} \right) \left( \frac{I}{Q} \right) = .95I \)

c) \( Q = 3.6 - 2.1P^3 + 6.7 \ln I \)

Price Elasticity: \( \epsilon_p = \frac{\% \Delta Q}{\% \Delta P} = \left( \frac{\Delta Q}{\Delta P} \right) \left( \frac{P}{Q} \right) = -.63P^{-2} \left( \frac{P}{Q} \right) = -.63P^3 \)

Income Elasticity: \( \epsilon_i = \frac{\% \Delta Q}{\% \Delta I} = \left( \frac{\Delta Q}{\Delta I} \right) \left( \frac{1}{Q} \right) = 6.7 \left( \frac{1}{Q} \right) \)

d) \( Q = 4P^{-1.6}I^{.45} \)

Let’s write this in logs first…

\( \ln Q = \ln 4 - 1.6 \ln P + .45 \ln I \)

Price Elasticity: \( \epsilon_p = \frac{\% \Delta Q}{\% \Delta P} = \left( \frac{\Delta \ln Q}{\Delta \ln P} \right) = -1.6 \)
Income Elasticity:  
\[ \varepsilon_I = \frac{\% \Delta Q}{\% \Delta I} = \left( \frac{\Delta \ln Q}{\Delta \ln I} \right) 0.45 \]

e)  
\[ Q = 6e^{-0.75P + 0.05I} \]

Let’s write this in logs first…

\[ \ln Q = \ln 6 - 0.75P + 0.05I \]

Price Elasticity:  
\[ \varepsilon_p = \frac{\% \Delta Q}{\% \Delta P} = \left( \frac{\Delta \ln Q}{\Delta P} \right) P = -0.75P \]

Income Elasticity:  
\[ \varepsilon_I = \frac{\% \Delta Q}{\% \Delta I} = \left( \frac{\Delta \ln Q}{\Delta I} \right) I = 0.05I \]

2) Suppose that you are selling lightbulbs door to door. You purchased all your lightbulbs up front so your costs are currently all sunk. You are currently selling you lightbulbs for $3.50 apiece and you sell 25 lightbulbs per day. You know that the elasticity of demand for lightbulbs at your current price is -.6. Is your price too high or too low? Explain.

With your costs being sunk, maximizing revenues will also maximize your profits. We know that total revenues are

\[ TR = PQ \] (Revenues equal price times quantity – your current revenues are $87.50)

Write this in percentage terms.

\[ \% \Delta TR = \% \Delta P + \% \Delta Q \]

We know that elasticity gives us

\[ \varepsilon_p = \frac{\% \Delta Q}{\% \Delta P} \Rightarrow \% \Delta Q = \varepsilon \% \Delta P \]
So I can substitute in….

\[ \% \Delta TR = (1 + \varepsilon) \% \Delta P \]

We face an elasticity equal to -0.6

\[ \% \Delta TR = (A) \% \Delta P \]

So, a higher price will raise profits (if percentage change in P is positive, percentage change in TR is positive)

3) Suppose you know that you face the following demand curve:

\[ Q = 150 - 3P \]

Calculate the price that maximizes revenues.

\[ TR = PQ = 150P - 3P^2 \]

\[ \frac{\Delta TR}{\Delta P} = 150 - 6P = 0 \]

\[ P = 25 \]

4) Suppose that you are currently charging a price of $40. You know that at your current price, income elasticity is equal to 1.5 and price elasticity equals -2.5. If you see a 20% increase in income, calculate the price change required to maintain your current sales level.

We know that

\[ \% \Delta Q = \varepsilon_p \% \Delta P + \varepsilon_I \% \Delta I \]

Plug in what we know...

\[ 0 = -2.5 \% \Delta P + 1.5(20) \]

\[ \% \Delta P = 12 \]
5) Suppose that you have estimated the following demand curve for DVDs.

\[ Q = 280 - 4P - 80D \]

\[ D = \begin{cases} 
1, & \text{if over 60 years old} \\
0, & \text{if under 60 years old} 
\end{cases} \]

What price(s) would you charge to maximize revenues?

So, after substituting in the dummy variable, we have the following demand curve(s)

\[ Q = \begin{cases} 
200 - 4P, & \text{if over 60 years old} \\
280 - 4P, & \text{if under 60 years old} 
\end{cases} \]

For Over 60 years old:

\[ TR = PQ = 280P - 4P^2 \]

\[ \frac{\Delta TR}{\Delta P} = 200 - 8P = 0 \]

\[ P = 25 \]

For Over 60 years old:

\[ TR = PQ = 200P - 4P^2 \]

\[ \frac{\Delta TR}{\Delta P} = 280 - 8P = 0 \]

\[ P = 35 \]

6) Consider a consumer choosing between three goods.

\[ P = (P_1, P_2, P_3), \quad X = (X_1, X_2, X_3) \quad (\text{i.e. three prices, three products}) \]

Each of the following groups represents choices of X1, X2, and X3 for various prices of X1, X2, and X3. Determine which group is inconsistent with rational choice.
First, calculate the cost of each bundle at each set of prices:

<table>
<thead>
<tr>
<th>Choice A</th>
<th>Choice B</th>
<th>Choice C</th>
</tr>
</thead>
<tbody>
<tr>
<td>X = (3,2,1)</td>
<td>X = (2,2,1)</td>
<td>X = (1,2,1)</td>
</tr>
<tr>
<td>P = (1,2,3)</td>
<td>$10</td>
<td>$9</td>
</tr>
<tr>
<td>P = (2,1,2)</td>
<td>$10</td>
<td>$8</td>
</tr>
<tr>
<td>P = (3,5,1)</td>
<td>$20</td>
<td>$17</td>
</tr>
</tbody>
</table>

At P = (1,2,3), A was chosen when B and C were cheaper: A > B, A > C
At P = (2,1,2), B was chosen when C was cheaper: B > C
At P = (3,5,1), C was chosen, A and B were more expensive – no information

Everything here is consistent!
Again, calculate the cost of each bundle at each set of prices:

<table>
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<tr>
<th>Choice A</th>
<th>Choice B</th>
<th>Choice C</th>
</tr>
</thead>
<tbody>
<tr>
<td>X = (5,1,3)</td>
<td>X = (3,3,3)</td>
<td>X = (4,2,2)</td>
</tr>
<tr>
<td>P = (3,4,1)</td>
<td>$22</td>
<td>$24</td>
</tr>
<tr>
<td>P = (2,3,2)</td>
<td>$19</td>
<td>$21</td>
</tr>
<tr>
<td>P = (5,3,1)</td>
<td>$31</td>
<td>$28</td>
</tr>
</tbody>
</table>

At P = (3,1,4), A was chosen when C was the same price: A > C
At P = (2,3,2), B was chosen when both A and C were cheaper: B > A, B > C
At P = (5,3,1), C was chosen when B was the same price: C > B

We have a contradiction here. At one set of prices, B > C, but at another, C > B. This is inconsistent with rational behavior!!

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</tr>
<tr>
<td>P = (4,3,2)</td>
<td>$18</td>
<td>$19</td>
</tr>
<tr>
<td>P = (5,3,3)</td>
<td>$22</td>
<td>$23</td>
</tr>
<tr>
<td>P = (5,2,3)</td>
<td>$20</td>
<td>$20</td>
</tr>
</tbody>
</table>

At P = (4,3,2), A was chosen when C was cheaper: A > C
At P = (5,3,3), B was chosen when A and C were cheaper: B > C, B > A
At P = (5,3,2), C was chosen, A and B were more expensive – no information.

Everything is consistent here!
7) Suppose that you had preferences of the form:

\[ U(x, y) = (xy)^{\frac{1}{3}} \]

The price of \( x \) is $2 and the price of \( y \) is $4. You have $120 to spend over the upcoming 5 days. How would you spend your money over the 5 day period?

First, let’s figure out how much total consumption of \( x \) and \( y \) you would choose over the entire period.

The optimality condition is

\[ \frac{MU_x}{MU_y} = \frac{P_x}{P_y} \]

So, with our utility function....

\[ MRS = \frac{MU_x}{MU_y} = \frac{(1/3)x^{2/3}y^{1/3}}{(1/3)x^{1/3}y^{2/3}} = \frac{y}{x} \]

Now, the optimality condition...

\[ \frac{y}{x} = \frac{2}{4} = \frac{P_x}{P_y} \]

\[ x = 2y \]

So, you will consume twice as much \( x \) as \( y \) over the 5 day period. Using the budget constraint, we can find the total quantities of \( x \) and \( y \) consumed

\[ 2x + 4y = 120 \]
\[ 2(2y) + 4y = 120 \]
\[ 8y = 120 \]
\[ y = 15 \Rightarrow p_y y = 4(15) = 60 \]
\[ x = 30 \Rightarrow p_x x = 2(30) = 60 \]

So, you spend half your income on \( x \) and half on \( y \). Now, WHEN do you consume \( x \) and \( y \). First, consider the total utility from consuming it all on day 1.
Now, suppose that you consume half today, half tomorrow, and nothing for the last three days

\[ U(15, 7.5) + U(15, 7.5) = 2 \left(112.5\right)^{\frac{1}{3}} = 9.63 \]  

You are happier!

Keep this going...the best choice is to spread it out evenly over 5 days...

\[ 5U(6, 3) = 5 \left(18\right)^{\frac{1}{3}} = 13.09 \]

8) Now, suppose that you had preferences of the form:

\[ U(x, y) = (xy)^3 \]

The price of x is $2 and the price of y is $4. You have $120 to spend over the upcoming 5 days. How would you spend your money over the 5 day period?

If we recalculate the MRS, we get

\[ MRS = \frac{MU_x}{MU_y} = \frac{3x^2y^3}{3x^3y^2} = \frac{y}{x} \]

So, we know that the same amounts of x and y will be consumed over the 5 day period ...30 of x and 15 of y. Again, the question is WHEN. Compare consuming all at once vs. spreading it out evenly.

\[ U(30, 15) = \left(450\right)^{\frac{1}{3}} = 91,125,000 \]

\[ 5U(6, 3) = \left(18\right)^{\frac{1}{3}} = 5,832 \]

In this case, the best thing to do is binge on one of the five days and go without for the remaining 4!

9) Consider the following utility functions:

For each of the utility functions:
a) Calculate the marginal rate of substitution
b) Calculate the elasticity of substitution

\[ U(x, y) = ax + by \quad (\text{Linear}) \]

\[ MRS = \frac{MU_x}{MU_y} = \frac{a}{b} \]

\[ \varepsilon = \frac{\% \Delta \left( \frac{y}{x} \right)}{\% \Delta MRS} = \Delta \left( \frac{y}{x} \right) \left( \frac{MRS}{\frac{y}{x}} \right) = \infty \]

\[ U(x, y) = x^\alpha y^\beta \quad (\text{Cobb - Douglas}) \]

\[ MRS = \frac{MU_x}{MU_y} = \frac{\alpha x^{\alpha-1} y^\beta}{\beta x^\alpha y^{\beta-1}} = \left( \frac{\alpha}{b} \right) \left( \frac{y}{x} \right) \]

\[ \varepsilon = \frac{\% \Delta \left( \frac{y}{x} \right)}{\% \Delta MRS} = \Delta \left( \frac{y}{x} \right) \left( \frac{MRS}{\frac{y}{x}} \right) = 1 \]

\[ U(x, y) = \left( \alpha x^\rho + (1 - \alpha) y^\rho \right)^{\frac{1}{\rho}} \quad \rho \leq 1 \quad (\text{Constant Elasticity of Substitution}) \]

\[ MRS = \frac{MU_x}{MU_y} \]

With this function, we have

\[ MU_x = \left( x^\rho + y^\rho \right)^{\frac{1}{\rho}-1} (x^{\rho-1}) \]

\[ MU_y = \left( x^\rho + y^\rho \right)^{\frac{1}{\rho}-1} (y^{\rho-1}) \]

Therefore,
\[ MRS = \frac{MU_x}{MU_y} = \left(\frac{x}{y}\right)^{\rho - 1} = \left(\frac{y}{x}\right)^{1-\rho} \]

\[ \varepsilon = \frac{\%\Delta \left(\frac{y}{x}\right)}{\%\Delta MRS} = d\left(\frac{y}{x}\right)^{MRS} = \frac{1}{(1-\rho)\left(\frac{y}{x}\right)^{\rho - 1}} = \frac{1}{1-\rho} \]

10) Suppose that the price of good X is $4 and the price of good Y is $6. You have $100 to spend and your preferences over X and Y are defined as

\[ U(x, y) = x^3 y^3 \]

Solve for your optimal choice of X and Y.

I will solve this the long way first, and then the short way we have a problem of the form:

\[ \max_{x, y} f(x, y) \]

\[ \text{subject to } g(x, y) \geq 0 \]

we can set up the lagrangian as

\[ \ell = x^3 y^3 + \lambda (100 - 4x - 6y) \]

Taking the derivatives with respect to x and y and set equal to zero

\[ \frac{2}{3}x^{\frac{1}{3}}y^{\frac{1}{3}} - 4\lambda = 0 \]

\[ \frac{2}{5}x^{\frac{2}{3}}y^{\frac{2}{3}} - 6\lambda = 0 \]

Solving for lambda...
\[ \lambda = \frac{2x^{\frac{1}{3}} y^{\frac{1}{3}}}{12} \]

\[ \lambda = \frac{x^{\frac{2}{3}} y^{\frac{2}{3}}}{18} \]

Setting the above two equal to each other

\[ \frac{2x^{\frac{1}{3}} y^{\frac{1}{3}}}{12} = \frac{x^{\frac{2}{3}} y^{\frac{2}{3}}}{18} \]

If we simplify this down a bit:

\[ \frac{2x^{\frac{1}{3}} y^{\frac{1}{3}}}{12} = \frac{x^{\frac{2}{3}} y^{\frac{2}{3}}}{18} \quad (\text{Multiply both sides by 18}) \]

\[ 3x^{\frac{1}{3}} y^{\frac{1}{3}} = x^{\frac{2}{3}} y^{\frac{2}{3}} \quad (\text{Divide both sides by } x^{\frac{1}{3}}) \]

\[ 3y^{\frac{1}{3}} = xy^{\frac{2}{3}} \quad (\text{Divide both sides by } y^{\frac{2}{3}}) \]

\[ 3y = x \]

We could get to this point quicker by using the rule:

\[ MRS = \frac{MU_x}{MU_y} = \frac{P_x}{P_y} \]

Where

\[ U_x = 2\frac{x^{\frac{1}{3}} y^{\frac{1}{3}}}{3} \]

\[ U_y = \frac{x^{\frac{2}{3}} y^{\frac{2}{3}}}{3} \]

Now use the constraint to solve the rest of the problem.

\[ 3y = x \]
\[4(3y) + 6y = 100 \Rightarrow y = 5.55\]

\[4x + 6y = 100\]

\[Y = 5.55, \ X = 16.65\]

Note that all income is spent. \((4)(16.65) + (6)(5.55) = 100\). Further, note that you are spending \((2/3)\) of your income on \(X\), \((1/3)\) of your income on \(Y\).