Constrained Optimization: Step by Step

Most (if not all) economic decisions are the result of an optimization problem subject to one or a series of constraints:

- Consumers make decisions on what to buy constrained by the fact that their choice must be affordable.
- Firms make production decisions to maximize their profits subject to the constraint that they have limited production capacity.
- Households make decisions on how much to work/play with the constraint that there are only so many hours in the day.
- Firms minimize costs subject to the constraint that they have orders to fulfill.

All of these problem fall under the category of *constrained optimization*. Luckily, there is a uniform process that we can use to solve these problems. Here's a guide to help you out.

Maximizing Subject to a set of constraints:

$$\max_{x,y} f(x,y)$$
$$\textit{subject to} \quad g(\mathrm{x},\mathrm{y}) \geq 0$$

Step I: Set up the problem

Here's the hard part. We always want the problem structured in a particular way. Here, we are choosing to maximize $f(x,y)$ by choice of x and y. The function $g(x,y)$ represents a restriction or series of restrictions on our possible actions.

The setup for this problem is written as

$$\ell(x,y) = f(x,y) + \lambda g(x,y)$$

For example, a common economic problem is the consumer choice decision. Households are selecting consumption of various goods. However, consumers are not allowed to spend more than their income (otherwise they would buy infinite amounts of everything!!). Let's set up the consumer's problem:

Suppose that consumers are choosing between Apples (A) and Bananas (B). We have a utility function that describes levels of utility for every combination of Apples and Bananas.

$$A^{\frac{1}{2}}B^{\frac{1}{2}} = \text{Well being from consuming (A) Apples and (B) Bananas.}$$

Next we need a set pf prices. Suppose that Apples cost $4 apiece and Bananas cost $2 apiece. Further, assume that this consumer has $120 available to spend. They the income constraint is

$$\$2B + \$4A \leq \$120$$

However, they problem requires that the constraint be in the form $g(x, y) \geq 0$. In the above expression, subtract $2B and $4A from both sides. Now we have

$$0 \leq \underbrace{\$120 - \$2B - \$4A}_{g(A,B)}$$

Now, we can write out the *lagrangian*

$$\ell(A,B) = \underbrace{A^{\frac{1}{2}}B^{\frac{1}{2}}}_{f(A,B)} + \lambda\underbrace{(120 - 2B - 4A)}_{g(A,B)}$$

<u>Step II: Take the partial derivative with respect to each variable</u>

We have a function of two variables that we wish to maximize. Therefore, there will be two *first order conditions* (two partial derivatives that are set equal to zero).

In this case, our function is

$$\ell(A,B) = A^{\frac{1}{2}}B^{\frac{1}{2}} + \lambda(120 - 2B - 4A)$$

Take the derivative with respect to A (treating B as a constant) and then take the derivative with respect to B (treating A as a constant).

$$\ell_A(A,B) = \frac{1}{2}A^{-\frac{1}{2}}B^{\frac{1}{2}} - 4\lambda = 0$$
$$\ell_B(A,B) = \frac{1}{2}A^{\frac{1}{2}}B^{-\frac{1}{2}} - 2\lambda = 0$$

III: Solve the First order conditions for lambda

If we solve the above two equations for λ we get

$$\lambda = \frac{A^{-\frac{1}{2}}B^{\frac{1}{2}}}{8}$$
$$\lambda = \frac{A^{\frac{1}{2}}B^{-\frac{1}{2}}}{4}$$

IV: Set the two expressions for lambda equal to each other

$$\frac{A^{-\frac{1}{2}}B^{\frac{1}{2}}}{8} = \frac{A^{\frac{1}{2}}B^{-\frac{1}{2}}}{4}$$

If we simplify this down a bit:

$$\frac{A^{-\frac{1}{2}}B^{\frac{1}{2}}}{8} = \frac{A^{\frac{1}{2}}B^{-\frac{1}{2}}}{4}$$ (Multiply both sides by 8)

$$A^{-\frac{1}{2}}B^{\frac{1}{2}} = \frac{8A^{\frac{1}{2}}B^{-\frac{1}{2}}}{4}$$ (Divide both sides by $B^{-\frac{1}{2}}$)

$$A^{-\frac{1}{2}}B = \frac{8A^{\frac{1}{2}}}{4}$$ (Divide both sides by $A^{-\frac{1}{2}}$)

$$B = \frac{8A}{4}$$ (simplify)

$$B = 2A$$

This tells us that if we are acting optimally, we should always buy twice as many bananas as apples (which makes sense because they cost twice as much!). At this step, we should always have an expression that relates one variable to the other.

<u>V: Use the constraint to solve for the two variables separately</u>

Next, notice that the income constraint will always be met with equality (utility always increases as we buy more and more). Therefore, we know

$$2B + 4A = 120$$

We can use these to solve the rest of the problem.

$$\left.\begin{array}{l} B = 2A \\ \\ 2B + 4A = 120 \end{array}\right\} \quad 2(2A) + 4A = 120 \Rightarrow 8A = 120 \Rightarrow A = 15$$

We now know that we will buy 15 Apples and 30 Bananas (B = 2A). Notice that the income constraint is satisfied.

$$\$2(30) + \$4(15) = \$120$$

Minimizing Subject to a set of constraints:

$$\begin{array}{l} \min\limits_{x,y} f(x,y) \\ \textit{subject to} \quad g(\mathrm{x},\mathrm{y}) \geq 0 \end{array}$$

<u>Step I: Set up the problem</u>

This basically works the same way as the problem above. Here, we are choosing to **minimize** $f(x,y)$ by choice of x and y. The function $g(x,y)$ represents a restriction or series of restrictions on our possible actions.

The setup for this problem is written as

$$\ell(x,y) = f(x,y) - \lambda g(x,y)$$

Note that the setup is identical with the exception that the **second term in the above expression is being subtracted rather than added.**

The usual problem is a firm trying to minimize costs subject to the requirement that it must produce a certain amount of output.

Suppose that a firm is choosing levels of labor and capital (L and K). Output is produced according to the following process

$K^{\frac{1}{2}}L^{\frac{1}{2}}$ = Firm Output (I chose the same function as above to simplify things)

Next we need a set of prices. Suppose that units of capital cost \$3 apiece and hours of labor cost \$9. We can write out total costs for the firm as the sum of capital costs and labor costs.

$$Total\ Costs = \$3K + \$9L$$

The firm wants to minimize the total costs of producing (at least) 100 units of output.

$$K^{\frac{1}{2}}L^{\frac{1}{2}} \geq 100$$

Therefore, the problem we face is

$$\min_{x,y}\{3K + 9L\}$$

$$subject\ to \quad K^{\frac{1}{2}}L^{\frac{1}{2}} \geq 100$$

Again, the problem requires that the constraint be in the form $g(x, y) \geq 0$. In the above expression, subtract 100 from both sides. Now we have

$$\underbrace{K^{\frac{1}{2}}L^{\frac{1}{2}} - 100}_{g(K,L)} \geq 0$$

We can write the problem as

$$\ell(x, y) = 3K + 9L - \lambda\left(K^{\frac{1}{2}}L^{\frac{1}{2}} - 100\right)$$

Again, note that the setup is identical with the exception that the **second term in the above expression is being subtracted rather than added.**

<u>Step II: Take the partial derivative with respect to each variable</u>

We have a function of two variables that we wish to maximize. Therefore, there will be two *first order conditions* (two partial derivatives that are set equal to zero).

In this case, our function is

$$\ell(x,y) = 3K + 9L - \lambda\left(K^{\frac{1}{2}}L^{\frac{1}{2}} - 100\right)$$

Take the derivative with respect to A (treating B as a constant) and then take the derivative with respect to B (treating A as a constant).

$$\ell_K(A,B) = 3 - \lambda\left(\frac{1}{2}K^{-\frac{1}{2}}L^{\frac{1}{2}}\right) = 0$$

$$\ell_L(K,L) = 9 - \lambda\left(\frac{1}{2}K^{\frac{1}{2}}L^{-\frac{1}{2}}\right) = 0$$

<u>III: Solve the First order conditions for lambda</u>

If we solve the above two equations for λ we get

$$\lambda = \frac{3}{\frac{1}{2}K^{-\frac{1}{2}}L^{\frac{1}{2}}}$$

$$\lambda = \frac{9}{\frac{1}{2}K^{\frac{1}{2}}L^{-\frac{1}{2}}}$$

<u>IV: Set the two expressions for lambda equal to each other</u>

$$\frac{3}{\frac{1}{2}K^{-\frac{1}{2}}L^{\frac{1}{2}}} = \frac{9}{\frac{1}{2}K^{\frac{1}{2}}L^{-\frac{1}{2}}}$$

If we simplify this down a bit:

$$\frac{3}{\frac{1}{2}K^{-\frac{1}{2}}L^{\frac{1}{2}}} = \frac{9}{\frac{1}{2}K^{\frac{1}{2}}L^{-\frac{1}{2}}} \quad \text{(Cancel out the 1/2s and divide both sides by 3)}$$

$$\frac{1}{K^{-\frac{1}{2}}L^{\frac{1}{2}}} = \frac{9}{3K^{\frac{1}{2}}L^{-\frac{1}{2}}} \qquad \text{(Multiply both sides by } K^{\frac{1}{2}}\text{)}$$

$$\frac{K}{L^{\frac{1}{2}}} = \frac{9}{3L^{-\frac{1}{2}}} \qquad \text{(Multiply both sides by } L^{-\frac{1}{2}}\text{)}$$

$$\frac{K}{L} = \frac{9}{3} \qquad \text{(simplify)}$$

$$\frac{K}{L} = 3$$

This tells us that if we are acting optimally, we should always employ three times as much capital as labor (which makes sense because labor costs three times as much!). At this step, we should always have an expression that relates one variable to the other.

V: Use the constraint to solve for the two variables separately

Next, notice that the production constraint will always be met with equality (your costs will always go down if you produce less). Therefore, we know

$$K^{\frac{1}{2}}L^{\frac{1}{2}} = 100$$

We can use these to solve the rest of the problem.

$$\left.\begin{aligned} K &= 3L \\ K^{\frac{1}{2}}L^{\frac{1}{2}} &= 100 \end{aligned}\right\} \quad (3L)^{\frac{1}{2}}L^{\frac{1}{2}} = 100 \Rightarrow (3)^{\frac{1}{2}}L = 100 \Rightarrow L = \frac{100}{\sqrt{3}} = 57.8$$

Therefore, L = 57.8, K = 3L = 173.41, and (if you check, total production does equal 100)