Finance 30210  
Problem Set #5

1) For each of the following demand curves, calculate the price elasticity of demand and the income elasticity of demand.

a) \( Q = 800 - 4P + 2I \)

Price Elasticity: \( \varepsilon_p = \frac{\%\Delta Q}{\%\Delta P} = \frac{\Delta Q}{\Delta P} \left( \frac{P}{Q} \right) = -4 \left( \frac{P}{Q} \right) \)

Income Elasticity: \( \varepsilon_I = \frac{\%\Delta Q}{\%\Delta I} = \frac{\Delta Q}{\Delta I} \left( \frac{I}{Q} \right) = 2 \left( \frac{I}{Q} \right) \)

b) \( \ln Q = 2.5 - .65 \ln P + .95I \)

Price Elasticity: \( \varepsilon_p = \frac{\%\Delta Q}{\%\Delta P} = \frac{\Delta Q}{\Delta \ln P} = - .65 \)

Income Elasticity: \( \varepsilon_I = \frac{\%\Delta Q}{\%\Delta I} = \frac{\Delta Q}{\Delta \ln I} \left( \frac{I}{Q} \right) = .95I \)

c) \( Q = 3.6 - 2.1P^3 + 6.7 \ln I \)

Price Elasticity: \( \varepsilon_p = \frac{\%\Delta Q}{\%\Delta P} = \frac{\Delta Q}{\Delta P} \left( \frac{P}{Q} \right) = - .63P^{-3} \left( \frac{P}{Q} \right) = - \frac{.63P^3}{Q} \)

Income Elasticity: \( \varepsilon_I = \frac{\%\Delta Q}{\%\Delta I} = \frac{\Delta Q}{\Delta \ln I} \left( \frac{1}{Q} \right) = \frac{6.7}{Q} \)

d) \( Q = 4P^{-1.6} I^{.45} \)

Let’s write this in logs first...

\( \ln Q = \ln 4 - 1.6 \ln P + .45 \ln I \)

Price Elasticity: \( \varepsilon_p = \frac{\%\Delta Q}{\%\Delta P} = \frac{\Delta \ln Q}{\Delta \ln P} = -1.6 \)
Income Elasticity:  \[ e_i = \frac{\% \Delta Q}{\% \Delta I} = \left( \frac{\Delta \ln Q}{\Delta \ln I} \right) .45 \]

c)  \[ Q = 6e^{-75P + .05I} \]

Let’s write this in logs first…

\[ \ln Q = \ln 6 - .75P + .05I \]

Price Elasticity:  \[ \epsilon_p = \frac{\% \Delta Q}{\% \Delta P} = \left( \frac{\Delta \ln Q}{\Delta P} \right) P = -.75P \]

Income Elasticity:  \[ \epsilon_i = \frac{\% \Delta Q}{\% \Delta I} = \left( \frac{\Delta \ln Q}{\Delta I} \right) I = .05I \]

2) Consider a consumer choosing between three goods.

\[ P = (P_1, P_2, P_3), \ X = (X_1, X_2, X_3) \ (i.e. \ three \ prices, \ three \ products) \]

Each of the following groups represents choices of X1, X2, and X3 for various prices of X1, X2, and X3. Determine which group is inconsistent with rational choice.

First, calculate the cost of each bundle at each set of prices:

<table>
<thead>
<tr>
<th></th>
<th>Choice A</th>
<th>Choice B</th>
<th>Choice C</th>
</tr>
</thead>
<tbody>
<tr>
<td>X = (3,2,1)</td>
<td>$10</td>
<td>$9</td>
<td>$8</td>
</tr>
<tr>
<td>P = (1,2,3)</td>
<td>$10</td>
<td>$8</td>
<td>$6</td>
</tr>
<tr>
<td>P = (2,1,2)</td>
<td>$20</td>
<td>$17</td>
<td>$14</td>
</tr>
<tr>
<td>P = (3,5,1)</td>
<td>$20</td>
<td>$17</td>
<td>$14</td>
</tr>
</tbody>
</table>

At P = (1,2,3), A was chosen when B and C were cheaper: A>B, A>C
At P = (2,1,2), B was chosen when C was cheaper: B>C

First, calculate the cost of each bundle at each set of prices:
At $P = (3,1,4)$, A was chosen when C was the same price: $A > C$
At $P = (2,3,2)$, B was chosen when both A and C were cheaper: $B > A$, $B > C$
At $P = (5,3,1)$, C was chosen when B was the same price: $C > B$

We have a contradiction here. At one set of prices, $B > C$, but at another, $C > B$. This is inconsistent with rational behavior!!

3) Suppose you know that you face the following demand curve:

$$Q = 150 - 3P$$

Calculate the price that maximizes revenues.

$$TR = PQ = 150P - 3P^2$$

$$\frac{\Delta TR}{\Delta P} = 150 - 6P = 0$$

$$P = 25$$
4) Suppose that you are currently charging a price of $40. You know that at your current price, income elasticity is equal to 1.5 and price elasticity equals -2.5. If you see a 20% increase in income, calculate the price change required to maintain your current sales level.

We know that

\[ \%\Delta Q = \varepsilon_p \%\Delta P + \varepsilon_i \%\Delta I \]

Plug in what we know…

\[ 0 = -2.5\%\Delta P + 1.5(20) \]

\[ \%\Delta P = 12 \]

5) Consider the following utility functions:

For each of the utility functions:

a) Calculate the marginal rate of substitution

b) Calculate the elasticity of substitution

\[ U(x, y) = ax + by \quad (\text{Linear}) \]

\[ MRS = \frac{MU_x}{MU_y} = \frac{a}{b} \]

\[ \varepsilon = \frac{\%\Delta \left( \frac{y}{x} \right)}{\%\Delta MRS} = \frac{\Delta \left( \frac{y}{x} \right)}{\Delta MRS} \left( \frac{MRS}{\frac{y}{x}} \right) = \infty \]

\[ U(x, y) = x^\alpha y^\beta \quad (\text{Cobb-Douglas}) \]

\[ MRS = \frac{MU_x}{MU_y} = \frac{\alpha x^{\alpha-1}y^\beta}{\beta x^\alpha y^{\beta-1}} = \left( \frac{\alpha}{\beta} \right) \left( \frac{y}{x} \right) \]
\[
\varepsilon = \frac{\% \Delta \left( \frac{y}{x} \right)}{\% \Delta MRS} = \frac{\Delta \left( \frac{y}{x} \right)}{\Delta MRS} \left( \frac{MRS}{\frac{y}{x}} \right) = 1 \left( \frac{\alpha}{\beta} \right) \left( \frac{y}{x} \right) = 1
\]

\[
U(x, y) = \left( \alpha x^\rho + (1 - \alpha) y^\rho \right)^{\frac{1}{\rho}} \quad \rho \leq 1 \quad \text{(Constant Elasticity of Substitution)}
\]

\[
MRS = \frac{MU_x}{MU_y}
\]

With this function, we have

\[
MU_x = \left( x^\rho + y^\rho \right)^{\frac{1}{\rho}} (x^{\rho - 1})
\]

\[
MU_y = \left( x^\rho + y^\rho \right)^{\frac{1}{\rho}} (y^{\rho - 1})
\]

Therefore,

\[
MRS = \frac{MU_x}{MU_y} = \left( \frac{x}{y} \right)^{\rho - 1} = \left( \frac{y}{x} \right)^{1 - \rho}
\]

\[
\varepsilon = \frac{\% \Delta \left( \frac{y}{x} \right)}{\% \Delta MRS} = \frac{d \left( \frac{y}{x} \right) MRS}{d MRS} = \frac{1}{1 - \rho} \left( \frac{y}{x} \right)^{1 - \rho} = 1
\]

6) Suppose that the price of good X is $4 and the price of good Y is $6. You have $100 to spend and your preferences over X and Y are defined as

\[
U(x, y) = x^2 \frac{1}{3} y^3
\]

Solve for your optimal choice of X and Y.
I will solve this the long way first, and then the short way we have a problem of the form:

$$\max_{x,y} f(x, y)$$

subject to \( g(x, y) \geq 0 \)

Where \( f(x, y) = x^3 y^3 \) and \( g(x, y) = 100 - 4x - 6y \). Therefore, we can set up the lagrangian as

$$\ell(x, y) = x^3 y^3 + \lambda(100 - 4x - 6y)$$

Taking the first order conditions

$$\ell_x(x, y) = \frac{2}{3} x^{\frac{1}{3}} y^{\frac{1}{3}} - 4\lambda = 0$$

$$\ell_y(x, y) = \frac{1}{3} x^3 y^{-\frac{2}{3}} - 6\lambda = 0$$

Solving for lambda...

$$\lambda = \frac{2x^{\frac{1}{3}} y^{\frac{1}{3}}}{12}$$

$$\lambda = \frac{x^3 y^{-\frac{2}{3}}}{18}$$

Setting the above two equal to each other

$$\frac{2x^{\frac{1}{3}} y^{\frac{1}{3}}}{12} = \frac{x^3 y^{-\frac{2}{3}}}{18}$$

If we simplify this down a bit:

$$\frac{2x^{\frac{1}{3}} y^{\frac{1}{3}}}{12} = \frac{x^3 y^{-\frac{2}{3}}}{18} \quad \text{(Multiply both sides by 18)}$$
\[3x^3 y^1 = x^2 y^\frac{2}{3}\] (Divide both sides by \(x^3\))

\[3y^3 = xy^\frac{2}{3}\] (Divide both sides by \(y^3\))

\[3y = x\]

We could get to this point quicker by using the rule:

\[\text{MRS} = \frac{U_x}{U_y} = \frac{P_x}{P_y}\]

Where

\[U_x = \frac{2}{3}x^\frac{1}{3}y^3\]

\[U_y = \frac{1}{3}x^2y^\frac{2}{3}\]

Now use the constraint to solve the rest of the problem.

\[3y = x\]

\[4(3y) + 6y = 100 \Rightarrow y = 5.55\]

\[4x + 6y = 100\]

\[Y = 5.55, X = 16.65\]

Note that all income is spent. \((\$4)(16.65) + (\$6)(5.55) = \$100\). Further, note that you are spending \((2/3)\) of your income on \(X\), \((1/3)\) of your income on \(Y\).