

MBA PROGRAM
UNIVERSITY OF NOTRE DAME

MATH WORKSHOP
AUGUST 13-20, 2004

Prof. Rick Mendenhall

Friday	Saturday	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday
Aug 13	Aug 14	Aug 15	Aug 16	Aug 17	Aug 18	Aug 19	Aug 20
2:30pm	no class	no class	9:00 am	9:00 am	9:00 am	9:00 am	8:45 am
5:00pm			11:30 am	11:30 am	11:30 am	11:30 am	11:30 am
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I. ALGEBRA

A. Solving a Linear Equation With One Unknown

A linear equation is one where the unknown appears only without an exponent. That is, X^2 , X^3 , etc. do not appear. Solving a linear equation for X is nothing more than a logical, sequential application of the four basic arithmetic operators—addition, subtraction, multiplication, and division with the goal of isolating X on one side of the equation.

Example: $7X + 2 = 4X + 5$ $X = ?$

Solution: Subtract $4X$ from each side: $3X + 2 = 5$

Subtract 2 from each side: $3X = 3$

Divide each side by 3: $X = 1$

Example: The price of a product is \$15. The fixed cost of production is \$500,000 and the variable cost is \$10 per unit. At what sales level (in units) will the product break even?

Solution: At breakeven $TR = TC$

$$TR = 15Q \quad TC = 500,000 + 10Q$$

$$15Q = 500,000 + 10Q$$

Subtract $10Q$ from each side

$$5Q = 500,000$$

Divide by 5

$$Q = 100,000 \text{ units}$$

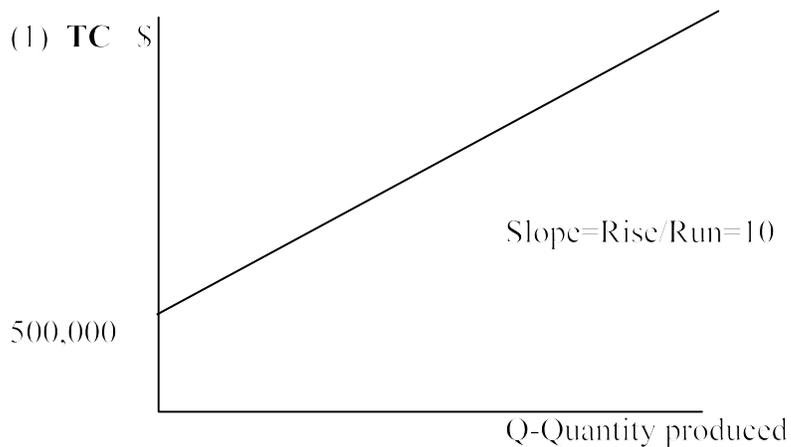
B. Straight Lines

We should be able to go from the equation of a line to a graph and vice versa.

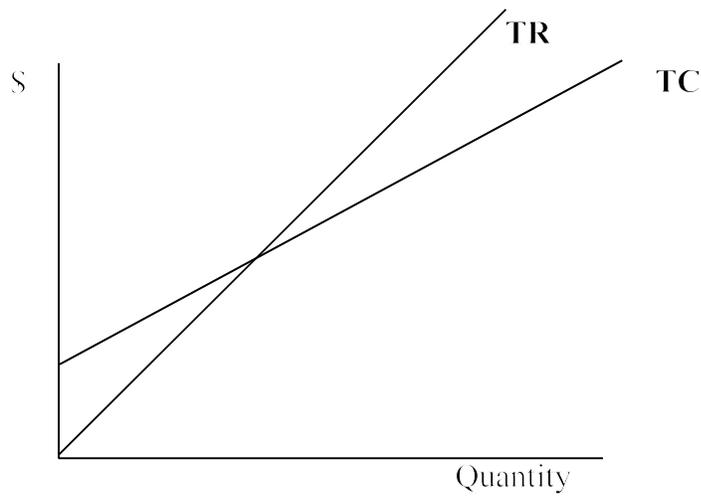
Example: From the last section, we are considering building and selling a product that will cost \$10 per unit to build. We will incur \$500,000 per year in fixed costs if we decide to introduce the product.

- (1) What does a plot of the TC equation look like?
- (2) What is the total cost as a function of Q?
- (3) What is total cost for $Q = 200,000$?
- (4) If the price is \$15 per unit, plot and calculate the breakeven volume?

Solution:



- (2) $y = m x + b$. m is the slope and b is the y intercept. x is the variable on the horizontal axis and y is the variable on the vertical axis. So,
 $TC = 10Q + 500,000$
- (3) $TC = 10Q + 500,000$
For $Q = 200,000$
 $TC = 10 (200,000) + 500,000 = \$2,500,000$
- (4) At breakeven $TC = TR$.



So, $10 Q +$

$$500,000 = 15 Q \Rightarrow 5Q = 500,000 \Rightarrow Q = 100,000 \text{ units}$$

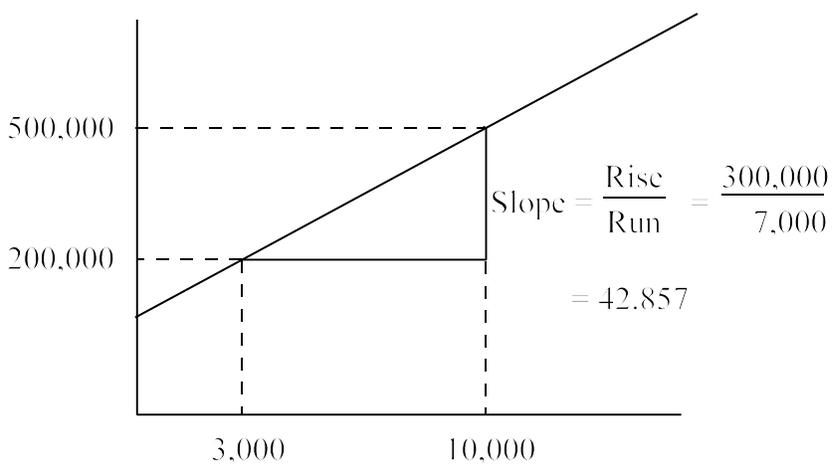
If I know any two points on a straight line, I can solve for the equation of the line.

Example:

For a different product:

At $Q = 10,000$, $TC = 500,000$ and at $Q = 3,000$, $TC = 200,000$

(1) Draw a picture of TC as a function of Q and solve for the equation.



Ok, the slope is 42.857, but what about the intercept, that is, what is the FC?

Simply choose either point, stick in the slope, and solve for b.

For example, choose $Q = 3,000$ and $TC = 200,000$.

$$Y = m X + b$$

$$TC = 42.857 Q + b$$

$$200,000 = 42.857 (3,000) + b$$

$$200,000 = 128,571 + b$$

$$b = 200,000 - 128,571 = 71,429$$

So, the equation for the line is $TC = 42.857 Q + 71,429$

Example: Economics courses often introduce a “budget line.” To demonstrate the budget line imagine that two goods or services, e.g., rice and beans, are available. Let y represent the units (e.g., pounds) of rice purchased and x represent the units of beans purchased. Say that the cost of rice is \$2 per pound, while beans cost \$3 per pound. If we can spend only \$25 on beans and rice then we can express the budget line (or budget constraint) as:

$$2y + 3x = 25$$

Where $2y$ is the total spent on rice and $3x$ is the total spent on beans.

What is the slope and the y -intercept for this line? Let's put into the $y = mx + b$ form.

$$2y + 3x = 25$$

subtract $3x$ from each side $2y = -3x + 25$

divide by 2 $y = -1.5 x + 12.5$

So, the intercept is 12.5 and the slope is -1.5. The intercept means that if we buy only rice, then we can purchase 12.5 pounds. The slope means that for every additional pound of beans that we buy, we must cut our rice purchases by one and one-half pounds.

C. Fractions

Operations with fractions may be performed according to the following rules.

1) The value of a fraction remains the same if its numerator and denominator are both multiplied or divided by the same number provided the number is not zero.

Examples. $\frac{3}{4} = \frac{3 \cdot 2}{4 \cdot 2} = \frac{6}{8}$, $\frac{15}{18} = \frac{15 \div 3}{18 \div 3} = \frac{5}{6}$

2) Changing the sign of the numerator or denominator of a fraction changes the sign of the fraction.

Example. $\frac{-3}{5} = -\frac{3}{5} = \frac{3}{-5}$

3) Adding two fractions with a common denominator yields a fraction whose numerator is the sum of the numerators of the given fractions and whose denominator is the common denominator.

Example. $\frac{3}{5} + \frac{4}{5} = \frac{3 + 4}{5} = \frac{7}{5}$

4) To find the sum or difference of two fractions having different denominators write the fractions with a common denominator.

Example. $\frac{1}{4} + \frac{2}{3} = \frac{3}{12} + \frac{8}{12} = \frac{11}{12}$

To find a common denominator you may multiply the two denominators.

Example. $\frac{1}{7} + \frac{2}{5} = \frac{5}{5} \cdot \frac{1}{7} + \frac{7}{7} \cdot \frac{2}{5} = \frac{5}{35} + \frac{14}{35} = \frac{19}{35}$

Or you may find the Least Common Denominator (LCD) by *factoring* each of the denominators.

Example. $\frac{1}{9} + \frac{5}{24} = ?$

$$9 = 3 \cdot 3$$

$$24 = 2 \cdot 2 \cdot 2 \cdot 3$$

There are three 2's in the factorization of 24 and two 3's in the factorization of 9.

So, the LCD is $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 72$.

$$\frac{1}{9} + \frac{5}{24} = \frac{8}{8} \cdot \frac{1}{9} + \frac{3}{3} \cdot \frac{5}{24} = \frac{8}{72} + \frac{15}{72} = \frac{23}{72}$$

Example. $\frac{2}{x^2} + \frac{3}{2x} + \frac{x}{7} = ?$

Calculate the LCD

$$x^2 = x \cdot x \quad 2x = 2 \cdot x \quad 7 = 7$$

The LCD = $2 \cdot 7 \cdot x \cdot x = 14x^2$

$$\frac{2}{x^2} = \frac{14}{14} \frac{2}{x^2} = \frac{28}{14x^2}$$

$$\frac{3}{2x} = \frac{7x}{7x} \frac{3}{2x} = \frac{21x}{14x^2}$$

$$\frac{x}{7} = \frac{2x^2}{2x^2} \frac{x}{7} = \frac{2x^3}{14x^2}$$

$$\frac{2}{x^2} + \frac{3}{2x} + \frac{x}{7} = \frac{28}{14x^2} + \frac{21x}{14x^2} + \frac{2x^3}{14x^2} = \frac{2x^3 + 21x + 28}{14x^2}$$

5) The product of two fractions is a fraction whose numerator is the product of the numerators of the given fractions and whose denominator is the product of the denominators of the fractions.

Examples. $\frac{2}{3} \cdot \frac{4}{5} = \frac{2 \cdot 4}{3 \cdot 5} = \frac{8}{15}$, $\frac{3}{4} \cdot \frac{8}{9} = \frac{3 \cdot 8}{4 \cdot 9} = \frac{24}{36} = \frac{2}{3}$

6) The reciprocal of a fraction is a fraction whose numerator is the denominator of the given fraction and whose denominator is the numerator of the given fraction. Thus the reciprocal of 3 (i.e., 3/1) is 1/3. Similarly the reciprocals of 5/8 and -4/3 are 8/5 and 3/-4 or -3/4, respectively.

7) To divide two fractions, multiply the first by the reciprocal of the second.

D. Solving a System of Linear Equations

If you have as many linear equations as you have unknowns then you can solve for the unknowns. Say we have two unknowns, X and Y, and these two equations:

$$(1) \quad X - 3Y = 7$$

$$(2) \quad -3X + 4Y = -1$$

What are X and Y?

Solution:

(i) Substitution

Solve for one variable in terms of the other, then substitute the expression for that variable into the other equation.

$$\text{From (1)} \quad X = 3Y + 7$$

Substitute the expression for X into (2)

$$-3(3Y + 7) + 4Y = -1$$

Now we have *one* equation and *one* unknown

$$-9Y - 21 + 4Y = -1$$

$$-5Y = 20 \Rightarrow Y = -4 \quad \text{and} \quad X = 3Y + 7 = 3(-4) + 7 = -5$$

(ii) Gaussian Elimination

In this method, we multiply one of the equations by a constant and add it to or subtract it from the other equation in order to remove one of the variables.

$$(1) \quad X - 3Y = 7$$

$$(2) \quad -3X + 4Y = -1$$

Multiply (1) by 3 and add it to (2)

$$(1') \quad 3X - 9Y = 21$$

$$(2) \quad \underline{-3X + 4Y = -1}$$

$$\text{Sum} \quad -5Y = 20 \Rightarrow Y = -4 \quad \text{Then we can use either equation to solve for } x.$$

(iii) Cramer's Rule

Cramer's rule is another method for solving a system of linear equations. Its advantage is that it is probably simpler to use when there are several unknowns and several equations. The disadvantage is that it requires some knowledge of matrices or at least the "determinants" of

matrices. If you wish to know more about it, let me know and I will give you additional material.

One more Example:

Say that the supply and demand of a particular good are given by the following two equations:

Supply: $Q = 10 + 0.50 P$ (1)

Demand: $Q = 200 - 2 P$ (2)

Solve for P and Q using substitution and Gaussian elimination.

Solution:

(i) Substitution

In this case, we simply set the two equations equal to one another.

$$10 + 0.50 P = 200 - 2 P$$

$$2.50 P = 190$$

$$P = \frac{190}{2.50} = \$76$$

$$Q = 200 - 2 (76) = 48 \text{ units}$$

(ii) Gaussian Elimination

Subtract (1) from (2)

$$0 = 190 - 2.5 P \Rightarrow P = \$76 \quad \text{And again use either equation to solve for Q.}$$

E. Different Types of Linear Relationships

1) Summations

Say that a company lets X_1 be sales for January, X_2 be sales for February, etc. Then total sales for the year could be designated as Y , where

$$Y = X_1 + X_2 + X_3 + \dots + X_{12}$$

We could write this as

$$Y = \sum_{i=1}^{12} X_i$$

\sum is a summation sign that allows for simplicity in writing an equation of this nature. Say that this company has 3 salespeople and wants to sum their monthly sales to get total annual sales.

We can use 2 subscripts $X_{i,j} \equiv$ sales for month i and salesperson j and total annual sales could be designated as $Y = X_{1,1} + X_{1,2} + X_{1,3} + X_{2,1} + X_{2,2} + X_{2,3} + X_{3,1} + \dots + X_{12,1} + X_{12,2} + X_{12,3}$

or

$$y = \sum_{i=1}^{12} \sum_{j=1}^3 X_{ij}$$

2) Altman's Z-Statistic

Example: Altman's Z-statistic is a formula designed to predict the possibility of corporate bankruptcy of manufacturing firms. Using a statistical technique called "discriminant analysis," Ed Altman came up with this equation.

$$Z = 1.2 X_1 + 1.4 X_2 + 3.3 X_3 + 0.6 X_4 + 1.0 X_5$$

where

$$X_1 = \text{Working Capital} / \text{Total Assets (or TA)}$$

$$X_2 = \text{Retained earnings} / \text{TA}$$

$$X_3 = \text{EBIT} / \text{TA}$$

$$X_4 = \text{Market value of equity} / \text{Book value of debt}$$

$$X_5 = \text{Sales} / \text{TA}$$

$Z > 3.00 \Rightarrow$ very low probability of bankruptcy

$Z < 1.80 \Rightarrow$ significant probability of bankruptcy

$1.80 < Z < 3.00 \Rightarrow$ Gray area

<u>Example</u>	TA = 1000	EBIT = 40	
	WC = 10	MV eq. = 800	(Shares outstanding times Price)
	RE = 100	BV debt = 500	Sales = 200

$$X_1 = \frac{WC}{TA} = \frac{10}{1000} = .01$$

$$X_2 = \frac{RE}{TA} = \frac{100}{1000} = .10$$

$$X_3 = \frac{EBIT}{TA} = \frac{40}{1000} = .04$$

$$X_4 = \frac{MVEq}{BV} = \frac{800}{500} = 1.6$$

$$X_5 = \frac{Sales}{TA} = \frac{200}{1000} = .2$$

Sticking the values into the Z equation:

$$Z = 1.2 (.01) + 1.4 (.10) + 3.3 (.04) + 0.6 (1.6) + 1.0 (.2) = 0.012 + 0.14 + 0.132 + 0.96 + 0.20$$

$Z = 1.444$ This firm seems to have a very significant probability of bankruptcy.

For an exercise, calculate the Z-statistic of the company you are studying in the Accounting Workshop.

F. Exponents

To indicate that a number is multiplied times itself some number of times, we use exponents.

Example $X^4 = X \cdot X \cdot X \cdot X$

There are several rules dictating the use of exponents:

1. $X^n X^m = X^{n+m}$

Example: $2^2 \cdot 2^3 = (2 \cdot 2)(2 \cdot 2 \cdot 2) = 2^5$

2. $X^n/X^m = X^{n-m}$

Example: $2^5 / 2^2 = (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) / (2 \cdot 2)$ The 2 twos in the denominator cancel with two in the numerator. So, $2^5 / 2^2 = 2 \cdot 2 \cdot 2 = 2^3$

3. $X^{-m} = 1/X^m$

Example: See the explanation for 2. above.

4. $(X^n)^m = X^{nm}$

Example: $(2^2)^3 = (2^2)(2^2)(2^2) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6$

5. $(XY)^n = X^n Y^n$

Example: $(2 \cdot 3)^2 = (2 \cdot 3)(2 \cdot 3) = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 3^2$

6. $X^{1/n} = \text{the } n\text{th root of } X$

Example:

$$\begin{aligned}\sqrt[2]{2^2} &= \sqrt[2]{4} = 2 \\ (2^2)^{1/2} &= 2^{2 \cdot 1/2} = 2^1 = 2\end{aligned}$$

So, taking a number to the 1/n power is like taking the nth root of the number.

7. $X^0 = 1$

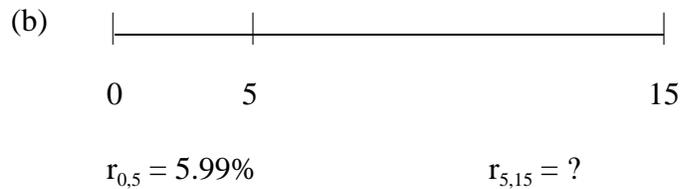
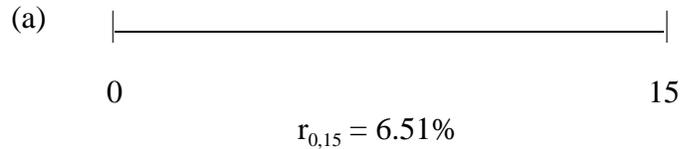
Proof: $X^m X^{-m} = X^{m-m} = X^0$ (from 1)

$X^m X^{-m} = X^m (1/X^m) = 1$ (from 2) So, $X^0 = 1$

Example: Under a theory called the “Expectations Hypothesis,” if one investor chooses to invest in one longer-term bond and another investor invests in a shorter-term bond and, upon maturity, reinvests the proceeds in another shorter-term bond, the two investors should expect to obtain the same long-term return

Example.

A 5-year bond is paying 5.99% per year. A 15-year bond is paying 6.51% per year. Under the expectations hypothesis, what is the expected annual rate of return on a 10-year bond 5 years from today?



$$(1 + r_{0,15})^{15} = (1 + r_{0,5})^5 (1 + r_{5,15})^{10}$$

$$\frac{(1 + r_{0,15})^{15}}{(1 + r_{0,5})^5} = (1 + r_{5,15})^{10}$$

$$(1 + r_{5,15}) = \left[\frac{(1 + r_{0,15})^{15}}{(1 + r_{0,5})^5} \right]^{1/10}$$

$$r_{5,15} = \left[\frac{(1 + r_{0,15})^{15}}{(1 + r_{0,5})^5} \right]^{1/10} - 1$$

$$r_{5,15} = \left[\frac{(1.0651)^{15}}{(1.0599)^5} \right]^{1/10} - 1$$

$$r_{5,15} = \left[\frac{2.57547}{1.33759} \right]^{1/10} - 1 = [1.92545]^{1/10} - 1$$

$$r_{5,15} = 1.06771 - 1 = .06771 = 6.771\%$$

(Non-math note: the expectations hypothesis is not believed to hold literally. Expectations regarding future short-term rates play an important role in determining long-term rates, but other factors play a role as well. These factors are discussed in other courses.)

G. The Exponential Function

There is a special “base” for exponents called “e” or the “exponential function.” e just happens to be approximately equal to 2.718281828, but many calculators have e buttons. e is special because “the rate of change of e^x is proportional to e^x .” In other words, e^x can be used to characterize the growth of something when the growth is proportional to how much of the something there is.

For example, e^x can be used to calculate how much money is in an account or how many people are in a particular area or are in the market for a particular product. In these cases, the growth rate, in terms of dollars per year or people per year, varies with the number of dollars invested or the number of people in the market. So, as the size of the account or the population grows, the rate of change (as measured in absolute terms such as dollars or people per year) grows as well.

Example. If money is being *continuously* compounded at a constant rate, the value at any future point in time is given by

$$V_t = V_0 e^{it}$$

where: V_t is the value at time t;

V_0 is the value at the present ($t = 0$);

i is the continuously compounded rate per unit time (e.g., year); and

t is the time from now (in, e.g., years).

If I invest \$1,000 in an account that compounds my money continuously at a rate of 6% per year, how much will I have in 5 years?

$$V_t = 1000 e^{(.06)(5)} = 1000 e^{.30} = \$1,349.86$$

Example. Some marketing studies suggest that if advertising and other promotions of a particular product are stopped, then at any time t the sales of that product will decline at a rate proportional to the amount of sales at time t.

If S_0 is the number of sales in the last month when advertising occurred and if $S(t)$ is the sales in the t^{th} month following month 0, then a reasonable model for $S(t)$ is

$$S(t) = S_0 e^{-\lambda t}$$

where λ is a positive number called the “sales decay constant.” The value of λ depends on the type of product, number of years of prior advertising, degree of competition, and other factors. Say that you estimate λ to be .2 and the current sales level is 10,000 units per year. What is the estimate for sales 10 years from now if you stop advertising?

$$S(10) = 10,000 e^{-(.20)10} = 10,000 e^{-2} = 1353 \text{ (rounded from 1353.35)}$$

H. Logarithms

Logarithms (or Logs) are the inverse functions to exponents. The definition of a logarithm is as follows.

If $b^X = N$ where $b > 0$, $b \neq 1$, and $N > 0$ then $X = \log_b N$.

“b” is known as the base. Almost any base ($b > 0$, $b \neq 1$) can be used but 2 are the most common: 10 and e.

Rules

$$(1) \quad \log_b MN = \log_b M + \log_b N$$

$$(2) \quad \log_b M/N = \log_b M - \log_b N$$

$$(3) \quad \log_b M^N = N \log_b M$$

When the base 10 is used, logs are called “common logs.” When e is used as the base (remember $e \approx 2.718281828$) logs are called “natural logs” and are often designated as Ln or ln.

A list of common logs

<u>Y</u>	<u>Log (Y)</u>	
0.10	-1	[Log (0.10) = -1 because $10^{-1} = 0.1$]
1	0	[$10^0 = 1$]
10	1	[$10^1 = 10$]
100	2	[$10^2 = 100$]
1,000	3	[$10^3 = 1,000$]
10,000	4	[$10^4 = 10,000$]

What are logs used for? There are some cases where logs allow us to solve problems in algebra and calculus that we would not be able to solve otherwise. Another common use of logs is to convert non-linear data into linear data. This is useful because most of the statistical techniques that we use assume linear relationships between the variables. In other words, it is easier to transform the data than to use a non-linear technique.

Example: $27^x = 43$. Solve for x.

Solution: $\ln(27^x) = \ln(43) \Rightarrow x \ln 27 = \ln 43$. I can now look up the ln's in log tables or use my calculator. I get $x(3.2958) = 3.7612 \Rightarrow x = 3.7612 / 3.2958 \Rightarrow 1.1412$. I could do the same thing with common logs.

Example: Ordinary Least Squares (OLS) regression, is a very common method of statistical analysis that you will see in many courses here and it is used extensively in the real world. It requires that you specify relationships between variables as linear. But say that a relationship is not linear, what can you do? One example is the reaction of stock prices to announcements such as earnings. It turns out that large firms exhibit smaller reactions per unit of surprise than small firms. But, it is not the case that an increase in size (market value of equity) from \$100 million to \$200 million has the same impact as going from \$1,000 million to \$1,100 million (each a change of \$100 million). The impact in the latter case is much smaller. If it is believed that doubling the size (e.g. from 100 million to 200 million *or* from \$1 billion to \$2 billion has approximately the same effect on the absolute magnitude of the return per unit of surprise, then converting the size figures to log of size (with any base) is appropriate. In fact that is what people do in this case.

Example: Here is another case using OLS. Say that you have a model that says the output of a production process can be specified as $Y = c X_1^A X_2^{1-A}$ (this is called a Cobb-Douglas production function), where C and A are constants and the X's are two inputs to the process (e.g., labor and capital for building cars or fertilizer and water for a grain harvest), and Y is the output (e.g., number of cars or bushels per acre). OLS cannot estimate the importance of the inputs in this form because of the exponents, but note:

$\ln Y = \ln (C X_1^A X_2^{1-A}) = \ln C + A \ln X_1 + (1-A) \ln X_2$. This is a linear equation and can be estimated using OLS regression.

II. CALCULUS

Derivatives

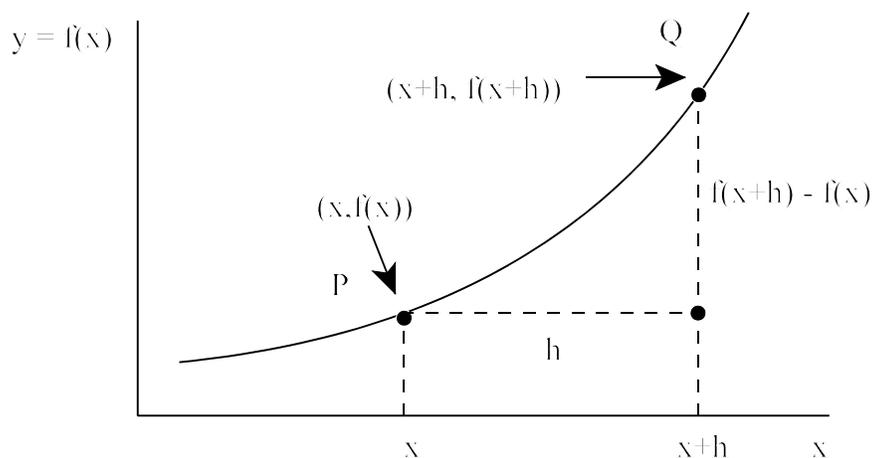
The derivative is a mathematical tool used to measure a rate of change. Rates of change may be visualized as the steepness of curves. The derivative provides a numerical measure of the steepness of a curve at a particular point.

The derivative has many applications in business as we'll see throughout this section. Just as a little foreshadowing, let me mention a few examples. The derivative of a total cost curve is the marginal or incremental cost of producing one more unit of a product. The derivative of total revenue is marginal revenue. For a division or company interested in maximizing profit, it is often the case that marginal cost is set equal to marginal revenue. Derivatives allow us to solve a whole host of optimization problems of this nature. If there is a cost to carrying inventory and a cost to placing each new order, how frequently should orders be placed? How frequently should machines be overhauled or forests harvested?

As stated above, derivatives find the steepness or slope of a curve at a point. The slope of a curve is defined as the slope of a straight line that is tangent to the curve at that point. Suppose that a curve is the graph of a function of x designated as $f(x)$. It is usually possible to obtain a formula that gives the slope of the curve $y = f(x)$ at a point. It is this formula that is called the derivative and it is designated as $f'(x)$. We sometimes say that $f'(x)$ is the derivative of the function $f(x)$ with respect to x . $f'(x)$ tells us how fast $f(x)$ is changing as x changes. The process of computing $f'(x)$ is called differentiation. There are several rules for differentiation that we will find very useful. At first, the process of differentiation may seem difficult, but it becomes easier with practice. Before we review these rules, let's discuss a general method of calculating derivatives that underlies many of them.

1. The Secant-Line Calculation of a Derivative

Consider a function, $f(x)$, for which we would like to calculate the derivative, and imagine two points, P and Q, on the graph of the function as shown below.



The slope of a straight line passing through P and Q is:

$$\text{Slope of line} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

Ok, this isn't the slope of $f(x)$ at point P (which is what we want), but as h grows very small, then the slope of the line will approach the slope of $f(x)$. That is, as Q moves close to P, the slope will get very close to $f'(x)$.

Summary:

To calculate $f'(x)$ using the secant-line approach:

1. Calculate $\frac{f(x+h) - f(x)}{h}$ (for $h \neq 0$).

2. Let h approach zero.

3. $\frac{f(x+h) - f(x)}{h}$ will approach $f'(x)$.

At this point you may have no clue as to what I'm talking about—hey,

none of this will be clear without examples. Let's try one.

Example: Consider $f(x) = x^2$. What is $f'(x)$? That is, can we obtain a formula that gives us the slope of the function x^2 at a point x ?

1. Calculate $\frac{f(x+h) - f(x)}{h}$

$$\frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = 2x+h$$

2. Let h approach zero. As h goes to zero $2x+h$ goes to $2x$.

3. $f'(x) = 2x$

It's true! The slope of the function $f(x) = x^2$ is equal to $2x$ at each point x . For example, the slope of x^2 at $x = 0$ is zero. At $x = 4$, the slope is 8 and at $x = 5$ the slope is 10. Ok, now some rules so we don't need to do this every time. The expression $d/dx [f(x)]$ is just another way of writing $f'(x)$. The rules sometimes involve two functions of x so $g(x)$ just represents the second function.

2. Rules for Differentiation

1. Constant multiple rule

K is a constant

$$\frac{d}{dx} [K f(x)] = K \frac{d}{dx} [f(x)]$$

2. Sum rule

$$\frac{d}{dx} [f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

3. Power rule

$$\frac{d}{dx} x^r = r x^{r-1}$$

4. General power rule

$$\frac{d}{dx} ([g(x)]^r) = r[g(x)]^{r-1} \cdot \frac{d}{dx} [g(x)]$$

5. Chain rule

$$y = f(u) \text{ and } u = g(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

6. Product Rule

$$\frac{d}{dx} [f(x) g(x)] = f(x) \frac{d}{dx} [g(x)] + g(x) \frac{d}{dx} [f(x)]$$

7. Quotient Rule

$$\frac{d}{dx} [f(x)/g(x)] = [g(x) \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x)]/[g(x)]^2$$

Example. Find the derivative of $f(x) = 3x^4$

Solution: From the power rule $d/dx (x^4) = 4x^3$, from the constant multiple rule, we just multiply $4x^3$ by 3 and $f'(x) = 12x^3$.

Example: Differentiate $f(x) = \frac{1}{(x^2-7)^5}$

Solution: $f(x) = \frac{1}{(x^2-7)^5} = (x^2-7)^{-5}$.

We have a function of x to a power. Let's use the general power rule.

$$f'(x) = -5(x^2-7)^{-6} (2x) = -10x(x^2-7)^{-6}$$

Example: Differentiate $(4x-1)(3x+1)^4$. What we have is the product of two functions of x , $(4x-1)$ and $(3x+1)^4$. The product rule tells us the derivative of this thing is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\text{Let } f(x) = (4x-1) \text{ and } g(x) = (3x+1)^4.$$

$$f'(x) = 4$$

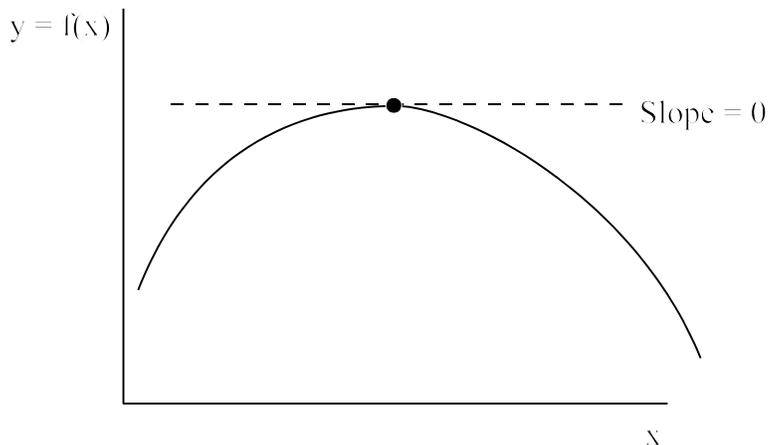
$$g'(x) = 4(3x+1)^3 (3) = 12(3x+1)^3 \text{ from the general power rule. So,}$$

$$\frac{d}{dx} [(4x-1)(3x+1)^4] = (4x-1) 12(3x+1)^3 + (4)(3x+1)^4$$

Well, can you interpret this? No? Neither can I. I know that this messy expression is the rate of change of our original expression of x or, in other words, if x goes up by one unit, then the original expression goes up by *about* $(4x-1) 12(3x+1)^3 + 4(3x+1)^4$ units. But what that looks like I don't really know. Let's do some applications. The functions tend to be much simpler and the result makes some logical sense.

3. Optimization Problems

Probably the most common application of derivatives for us will be optimization problems. In an optimization problem, we'll be either minimizing something (such as costs or material used), or maximizing something (such as profits or revenue). What is the derivative of a function that is being either maximized or minimized?



The slope is zero. So to optimize something, we will calculate the derivative, set it equal to zero and solve for x .

Example

The manager of a department store wants to build a 600-square-foot rectangular enclosure on the store's parking lot in order to display some equipment. Three sides of the enclosure will be built of redwood fencing, at a cost of \$14 per running foot. The fourth side will be built of cement blocks, at a cost of \$28 per running foot. Find the dimensions of the enclosure that will minimize the total cost of the building materials.

Solution: In order to solve this problem, we have to think a little bit. What do we know? (1) the length times the width must equal 600; and (2) we want to minimize the total cost. Let L = length and W = width. (Actually we don't need to specify that one is length and the other width.)

$$LW = 600$$

We are to have 3 sides at \$14/foot and one side at \$28/foot. The total cost can be expressed as

$$C = 14L + 14L + 14W + 28W = 28L + 42W$$

We want to minimize C with respect to either L or W , but we can't do either while both are in the equation. We must substitute for one of them. From $LW = 600$, we know that $L = 600/W$. Substitute this into the cost equation.

$$C = 28 \frac{600}{W} + 42W = \frac{16,800}{W} + 42W$$

All we need to do is take the derivative of C with respect to w , set it equal to zero and we'll have the optimal width.

$$C(W) = 16,800W^{-1} + 42W$$

$$C'(W) = -16,800W^{-2} + 42 = 0$$

$$42 = 16,800W^{-2}$$

$$\frac{42}{16,800} = W^{-2}$$

$$\frac{16,800}{42} = W^2$$

$$W = \left[\frac{16,800}{42} \right]^{1/2} = 20$$

(Technically, the answer is either 20 or -20, but we know that a wall cannot be -20 feet long.)

The optimal width appears to be 20 and the optimal length must be 30 (remember length times width equals 600).

This seems reasonable since the more expensive side is shorter. But when we set the derivative equal to zero, we might be maximizing a function we want to minimize or vice versa. How

would we know? First, with Excel, it is very easy to graph functions, so you can easily tell if you are maximizing or minimizing the function. In fact, you could use graphing as a check on your result or as an alternative to differentiation. But there are other ways to tell if you are maximizing or minimizing a function. One method is to take the second derivative of the function (denoted $f''(x)$). Just like $f(x)$ is a function of x , $f'(x)$ is a function of x and it can be differentiated. When you differentiate $f'(x)$, you get $f''(x)$ or the second derivative. What does $f''(x)$ tell you? $f''(x)$ tells you how $f'(x)$ is changing, just like $f'(x)$ tells you how $f(x)$ is changing. Now, if $f''(x)$ is less than zero at the value of x where $f'(x)$ is equal to zero, it means that the derivative is declining as x increases. This means you have found a maximum. If $f''(x)$ is positive at the x value where $f'(x)$ is equal to zero, then just the opposite—you have found a minimum. Let's check for this example.

$$C'(W) = 42 - 16,800W^{-2}$$

$$C''(W) = (-2)(-16,800)W^{-3} = 33,600W^{-3}$$

For any reasonable (positive) value of W , $C''(W)$ is positive. This means that the slope is increasing throughout. We have, indeed, found the point where cost is minimized.

Suggestions for Solving an Optimization Problem

1. Draw a picture, if possible.
2. Decide what quantity Q (cost, profits, etc.) is to be maximized or minimized.
3. Assign letters to other quantities that may vary.
4. Determine the “objective equation” that expresses Q as a function of the variables assigned in step 3.
5. Find the “constraint equation” that relates the variables to each other and to any constants that are given in the problem.
6. Use the constraint equation to simplify the objective equation in such a way that Q becomes a function of only one variable.
7. Differentiate Q , set it equal to zero, and solve for the decision variable.
Alternatively, sketch the graph of the function obtained in step 6 and use this graph to solve the optimization problem.

Example: A ship uses $5x^2$ dollars of fuel per hour at a speed of x miles per hour. The other expenses of operating the ship amount to \$2,000 per hour. What speed minimizes the cost of a 500-mile trip?

Solution: This problem really doesn't require all of the steps listed. As it turns out, all we have to do is properly specify the total cost function, take its derivative, set the derivative equal to zero, and solve for the speed, x . The total cost per hour is

$$5x^2 + 2000$$

But we're not concerned with total cost per hour, but rather total cost for a trip of 500 miles. If we travel 500 miles at speed x , how many hours will it take us?

$$\text{Distance} = (\text{speed}) (\text{time})$$

$$\text{time} = \frac{\text{Distance}}{\text{speed}}$$

In our case

$$\text{Hours} = \frac{500 \text{ miles}}{x \text{ miles/hour}}$$

So the total cost is cost per hour times number of hours.

$$C(x) = (5x^2 + 2000)\left(\frac{500}{x}\right) = 2500x + 1,000,000x^{-1}$$

$$C'(x) = 2500 + (-1) 1,000,000x^{-2} \quad (\text{By the power rule})$$

$$C'(x) = 0 = 2500 + (-1) 1,000,000x^{-2}$$

or

$$0 = 1 - 400x^{-2} \Rightarrow 400x^{-2} = 1 \Rightarrow$$

$$x^{-2} = \frac{1}{400}$$

$$x^2 = 400 \Rightarrow x = 20$$

The optimal speed is 20 miles per hour.

4. Partial Derivatives

Say that we have a function of two variables, $u = f(x,y)$. For example, the yield of a wheat crop (u) may depend on the amount of fertilizer used (x) and the level of rainfall (y). The partial derivative of u with respect to x is defined as the rate of change of u with a small unit change in x *while holding y constant*. So, to take the partial derivative of u with respect to x , designated $\partial u/\partial x$, we simply do it the same way as we have been and treat y as a constant.

Example:

Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x,y) = 5x^3y^2$

$$\frac{\partial f}{\partial x} = 5(3)x^2y^2 = 15x^2y^2$$

$$\frac{\partial f}{\partial y} = 5x^3(2)y = 10x^3y$$

In statistics, you will cover multiple regression. Multiple regression is a statistical technique that, among other things, is used to empirically estimate partial derivatives. That is, we could use it to estimate the additional bushels of wheat one acre will yield if we increase fertilizer usage by one pound while rainfall is held constant.

5. Lagrange Multipliers

Lagrange Multipliers provide a formal way of incorporating a constraint into an objective equation. While in some cases we could use the same optimization techniques as discussed above, it may be difficult to solve for one variable in terms of the other using the constraint equation. (E.g., if the constraint equation is something like $x^4 + 3x^3y + 5x^2y^4 + 12 = 0$.) Also, even if we can use the constraint equation to solve for one variable in terms of the other, substitution into the objective equation may result in a complicated function. We'll do a numerical example first and then use an application to economics as an in-class exercise.

Example:

$$\text{Maximize:} \quad 36 - x^2 - y^2$$

$$\text{Subject to the constraint:} \quad x + 7y - 25 = 0$$

Solution:

Form the auxiliary function of three variables:

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y) \quad \text{Where } f \text{ is the objective equation, } g \text{ is constraint equation and } \lambda \text{ is the Lagrange multiplier.}$$

$$F(x, y, \lambda) = 36 - x^2 - y^2 + \lambda (x + 7y - 25)$$

Take the partial derivative of $F(x, y, \lambda)$ with respect to each of the three variables and set each equation equal to zero.

$$\partial F / \partial x = -2x + \lambda = 0 \quad (1)$$

$$\partial F / \partial y = -2y + 7\lambda = 0 \quad (2)$$

$$\partial F / \partial \lambda = x + 7y - 25 = 0 \quad (3)$$

We now have three equations and three unknowns. We should be able to solve for each of the unknowns.

Use equations (1) and (2) to solve for λ in terms of x and in terms of y :

$$\text{From (1)} \quad \lambda = 2x \quad \text{From (2)} \quad \lambda = (2/7)y$$

$$2x = (2/7)y \quad \text{then} \quad x = (1/7)y$$

Substitute into (3):

$$(1/7)y + 7y - 25 = 0$$

$$(50/7)y = 25$$

$$y = 25(7/50) \quad \text{then} \quad y = 7/2$$

$$\text{If } y = 7/2, \text{ then } x = (1/7)(7/2) = 1/2$$

$$\text{If } x = 1/2, \text{ then } \lambda = 2(1/2) = 1$$

In-class exercise:

Suppose that x units of labor and y units of capital can produce $f(x,y) = 60 x^{3/4} y^{1/4}$ units of a certain product. One unit of labor costs \$100 and one unit of capital costs \$200. \$30,000 is available to invest in labor and capital. How many units of capital and how many units of labor should be used?

III. TIME VALUE OF MONEY and APPLICATIONS

In order to understand the financial value of just about anything, you must understand some principles that are collectively known as “the time value of money.” The financial value of anything is a function of several factors that include the size, timing, and riskiness of future cash flows, as well as current interest rates and the “price of risk.” (More about this in your finance course(s).)

Time value of money deals with size and timing of cash flows. The idea underlying TVM is that a dollar received today is not worth the same as a dollar received at some point in the future. Why? Because if I get the dollar today, I can start earning interest on it. The first step to learning the TVM is to understand the four basic operators.

A. The Four Operators

PVIF (n,i) - Present value factor

FVIF (n,i) - Future value factor

PVIFA (n,i) - Present value of annuity

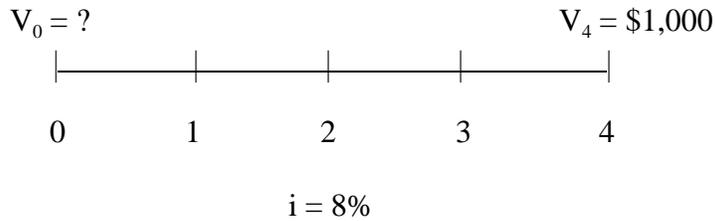
FVIFA (n,i) - Future value of annuity

$$\text{PVIF (n,i)} = (1+i)^{-n} \text{ or } \frac{1}{(1+i)^n}$$

“i” is the interest rate or required rate of return depending on the context and n is the number of periods. (We could just as easily use “r” for return, or “k” which is often used as required rate of return.)

Example:

If you wish to have \$1000 in an account in 4 years and the account pays 8% per year, how much must you invest today?



$$V_0 = 1000 \text{ PVIF}(4, 8\%)$$

$$= 1000 (1.08)^{-4} = \$735.03$$

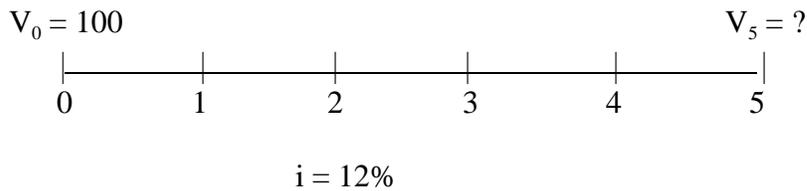
Another way to think of this problem is: “What is the value today of \$1000 to be received 4 years from today, if your *required rate of return* is 8%?” So, PVIF moves a cash flow’s value *back* in time.

FVIF (n,i) = (1 + i)ⁿ

Notice that FVIF is just the reciprocal of PVIF. FVIF moves a cash flow’s value *forward* in time.

Example:

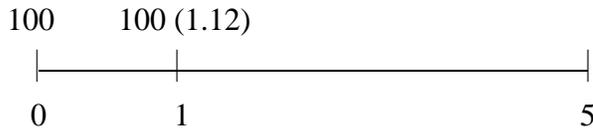
If I invest \$100 in an account that pays 12% per year, how much will be in my account in 5 years?



$$V_5 = 100 \text{ FVIF}(5, 12\%)$$

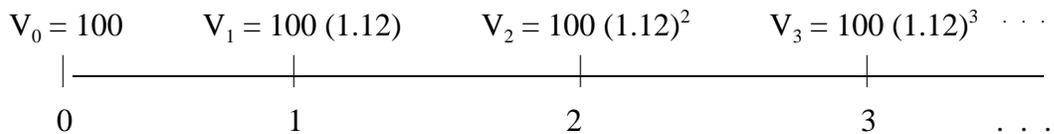
$$= 100 (1.12)^5 = \$176.23$$

It is easy to see why PVIF and FVIF are reciprocals. They do exactly opposite things. PVIF is used more in finance, but I think that FVIF is somewhat more intuitive. Think of the above problem one period at a time.



If I invest \$100 at a rate of 12%, then at the end of one year I will have my original \$100 plus I will have 12% of \$100. The return of my original \$100 is given by the 1 in 1.12 and the 12% is given by the .12.

Ok, so at the end of one year, I will have 100 (1.12) dollars. What if I invest this for another year at 12%. Again, I will have the amount invested at the start of the year plus 12%. So, I will multiply the beginning amount by another (1.12), etc.

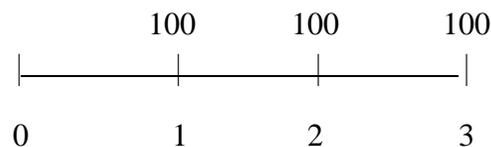


We say that PVIF and FVIF are used with "lump sums," single cash flows or single values. The next two operations are used with something called an annuity.

Annuities

An annuity is a set of equal cash flows that are equally spaced in time. A very common example of an annuity is a mortgage where a person borrows an amount of money and makes equal monthly payments for a fixed period.

Here is a picture of a simple annuity.



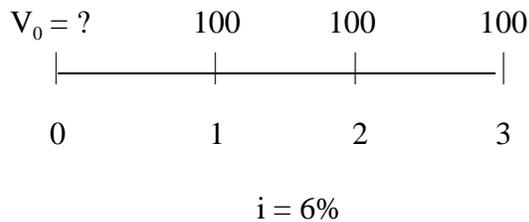
It is an annuity because the cash flows are equal (\$100) and they are equally spaced in time (one period apart). In order to value an annuity like this, let's consider a special type of annuity that goes on forever. Such an annuity is called a perpetual annuity or more commonly a *perpetuity*.

As you can imagine, working out this equation isn't too bad once or twice, but it would get tedious if we used it frequently. We can avoid the equation by using tables. The problem with tables is that they don't cover every possibility. For example, if $i = 6.93\%$, as may very well be the case in reality, you cannot use the table. You could interpolate, but the result will be a little wrong and interpolation is tedious. The third way to obtain the functions is to use a financial calculator.

Example:

Say that I want to find PVIFA (6, 8.43%) using my financial calculator. For my HP-12C and many other types, you can put in zero for "FV", 1 in for "pmt", 8.43 for "i" (or on some "int"), and 6 for "n." Press "PV" (or for some "CPT" then "PV") and PVIFA (6,8.43%) should pop up. It may appear as a negative number. Why?

Finally, before we go on to FVIFA, let me point something out. Consider this annuity.



- (1) $V_0 = 100 \text{ PVIFA}(3, 6\%)$
- (2) $V_0 = 100 \text{ PVIF}(1, 6\%) + 100 \text{ PVIF}(2, 6\%) + 100 \text{ PVIF}(3, 6\%)$

To solve this, I'll use the tables.

- (1) $V_0 = 100 (2.6730) = 267.30$
- (2) $V_0 = 100 (.9434) + 100 (.8900) + 100 (.8396) = 267.30$

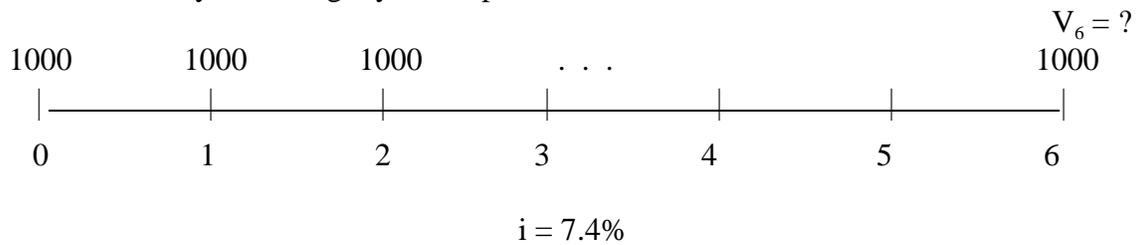
So, as we would suspect, using PVIFA once gives us the same result as bringing each cash flow back individually.

FVIFA (n,i) = ((1+i)ⁿ-1)/i

FVIFA does pretty much the same thing as PVIFA, but it finds the value of an annuity *at the time of the last payment*.

Example: An account pays 7.4%. If I invest \$1,000 into an account today and make \$1000 deposits at the end of each of the next 6 years (7 total payments), how much will be in my

account immediately following my last deposit?

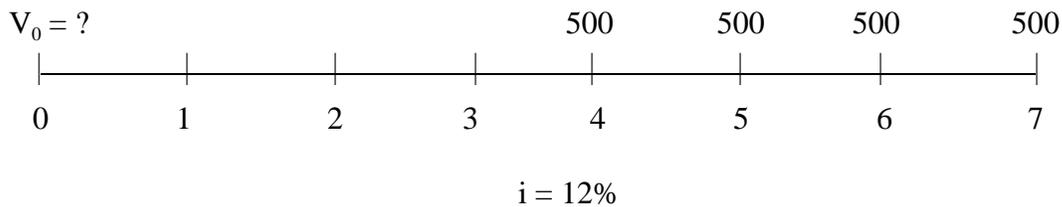


$$V_6 = 1000 \text{ FVIFA } (7, 7.4\%)$$

If I want to use my calculator to find FVIFA (7, 7.4%), I put in 7 for "n," 7.4 for "i" or "int," 1 for "pmt," and zero for "PV." Pushing "FV" should give me FVIFA (7, 7.4%). Doing the same thing, but putting 1000 in for "pmt" will solve the above problem.

Combining TVM operations

Example: Find the value at time 0.



Once we have used FVIFA or PVIFA to find the value of an annuity, we can then treat that annuity as a lump sum.

1. $V_3 = 500 \text{ PVIFA } (4, 12\%) = \1518.67
2. $V_0 = V_3 \text{ PVIF } (3, 12\%) = \1080.96

B. Different Compounding Intervals

The equations and calculators solve these problems in terms of periods (n) and rate per period (i). So far, we've almost always treated a period as a year, but it can be any length as long as we use the right interest rate. Clearly, if we were told the periods were months and the rate was on a per month basis, we could solve problems just as we have. But we can also solve problems when the compounding interval does not match the stated rate we are given.

At some time in the last few pages the thought may have occurred to you, “Hey, we used that ‘e’ thing to calculate a future value. What gives?” That’s right we did. The only difference is that for that example we said that the money was compounded *continuously*, unlike the annual compounding we’ve used so far. In most contexts there is a stated compounding interval such as annual, semi-annual, quarterly, monthly, etc., but compounding is sometimes continuous.) Let me point out that e^{in} is an FVIF and e^{-in} is a PVIF. Most interest rates are stated in annual terms, but a compounding interval different than one year is often specified. For example, a bank may offer a rate of 6.4% compounded semi-annually. The 6.4% is an annual rate, so the rate is really 3.2% every six months and every six months the interest is added to your account. In this case we might write FVIF (number of 6 six-month periods, 3.2%) = $(1 + .064/2)^{(years)(2)}$. More generally, consider FVIF when the money is compounded “m” times per year and we let “n” equal the number of years. $FVIF(nm, i/m) = (1 + i/m)^{nm}$. OK, where does e come in? $e^{in} = (1+i/m)^{nm}$ in the limit as m grows very large--continuous compounding.

(Explanatory note: I really don’t think of “n” as the number of years and “mn” as the number of periods. I just think of “n” as the number of periods and “i” as the rate per period. I used “m” above just to help explain the concept of different compounding intervals. For me, thinking about “m” when, for example, drawing a time line would just confuse things.)

Example: Equivalent Annual Rates (EARs)

There are three banks in town of equal safety. For short term accounts, bank A, S, and Q are offering 5.31% compounded annually, 5.15% compounded semi-annually, and 5.00% compounded quarterly, respectively. Which bank is offering the best rate?

To solve this problem we might calculate something called the "equivalent annual rate" or EAR. The EAR is the annually compounded rate that gives the same interest as a particular rate compounded at a non-annual interval. Bank A's stated interest is compounded annually, so we know that its EAR is 5.31%. We can use something I call the dollar trick to calculate the EARs for banks S and Q.

Bank S: A rate of 5.15% compounded semi-annually really means a rate of 5.15%/2 every 6 months.

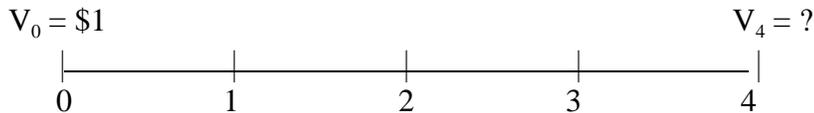


$$i = .0515/2 = .02575$$

If you invest \$1 in bank S for a year, you will end up with

$$V_2 = \$1 (1.02575)^2 = 1.05216$$

So, the rate of return must be $(1.05216-1)/1 = .05216$ or 5.216%. This is the EAR. We can do the same thing for bank Q.



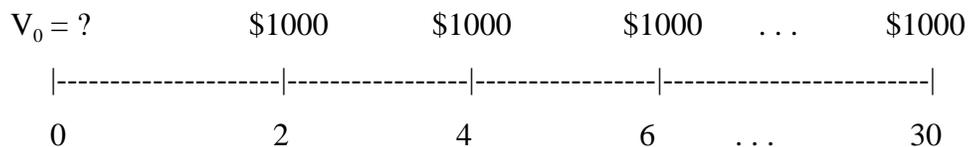
$$i = .0500/4 = .0125$$

$$V_4 = \$1 (1.0125)^4 = 1.0509$$

So, the EAR of bank Q is 5.09%. Bank A is the best deal.

In this example we took short term rates and found longer term rates. For example, for Bank Q we found that 1.25% per quarter is equivalent to 5.09% per year. Let's go the opposite direction now and turn two values over multiple years into an annualized rate of return.

Example: A person is offered a contract that allows them to receive \$1000 every two years for thirty years starting two years from today (i.e., first payment occurs at time 2, second payment at time 4, ..., fifteenth payment at time 30.) The investor's discount rate is 8% per year compounded annually. What is the value of the annuity today?



To value this as an annuity, we must find the effective rate between periods. If the rate for one year is 8%, then the effective rate for two years is given by the dollar trick. If I invest \$1 for two years at a rate of 8% per year compounded annually, how much money will I have?

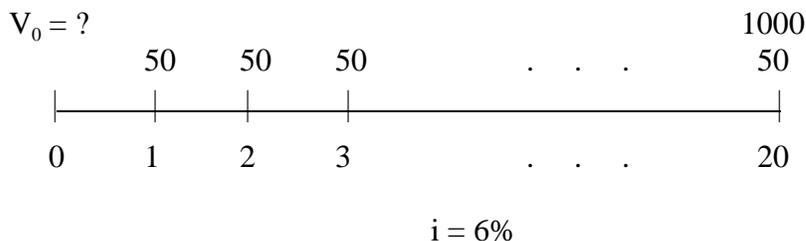
$X = 1 (1.08)^2 = 1.1664$. So, the effective two year rate must be 16.64%. The value of this annuity is, therefore, $V_0 = 1000 \text{ PVIFA} (15, 16.64\%) = \$5,412.40$.

C. Bond Valuation

If you understand the principles of TVM, then bond valuation, at least as much as we'll do, is quite simple. A bond represents debt. The issuer of the bond (e.g., a corporation, a municipality, a federal government) has borrowed money from the purchaser. So, a bond is nothing more than a contract between two parties that specifies the trading of a single cash flow at issuance (from the buyer of the bond to the issuer) for a set of cash flows in the future (from the issuer to the holder of the bond). Now, once the bond has been issued, it may be sold by the investor to another investor. As interest rates and/or the financial strength of the issuer changes, the price of the bond will change. Let's do an example. A typical bond in the U.S. consists of semi-annual interest payments until the bond "*matures*" and a lump sum payment at *maturity*. The lump sum payment is called the *par* or *face* value. The bond ceases to exist (the debt has been repaid) at the time of maturity. The bond contract specifies a *coupon rate*. This rate times the par value gives the annual interest the bond is to pay.

Example: A bond has just made an interest payment and has 10 years to maturity. The coupon rate is 10% and the required rate of return is 12% per year compounded semi-annually. Interest is paid semi-annually and the par value is \$1000. What is the price of the bond?

Annual interest is given by par value times coupon rate, which is 10% of \$1000 or \$100. Since interest is paid semi-annually, each payment is \$50. Since the required rate of return (or we could say yield-to-maturity) is 12% per year compounded semi-annually, the effective rate is 6% every six months. The bond contract can be represented by this picture.



We see the bond is nothing more than an annuity and a lump sum. The value today must be

$$V_0 = 50 \text{ PVIFA}(20, 6\%) + 1000 \text{ PVIF}(20, 6\%)$$

(Could we use 1000 PVIF (10, 12%)? Think about it.)

$$V_0 = 50 (11.4699) + 1000 (0.31180)$$

$$V_0 = 573.50 + 311.80 = 885.30$$

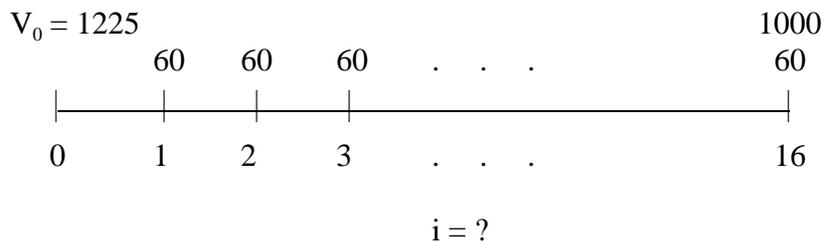
Given bond information, we can also solve for the yield-to-maturity (YTM). The YTM is the discount rate (the i) that sets the present value of the interest payments plus the present value of the par value equal to the current price. The YTM is a measure of the return an investor can expect to get on the bond. (The subtleties of exactly what YTM tells us and what it doesn't tell us are beyond the scope of this course.)

Example:

Maturity = 8 years
 Par value = \$1000
 Coupon rate = 12%
 Price = \$1,225
 Interest is semi-annual

What is the yield-to-maturity?

Solution:



We use exactly the same equation as before, but we put in the price and solve for i .

$$1225 = 60 \text{ PVIFA}(16, i\%) + 1000 \text{ PVIF}(16, i\%)$$

The i that solves this equation is the bond's YTM. Unfortunately, the only way to solve this is by trial and error. Since the price (\$1225) is larger than the par value (\$1000), I know that the

YTM is less than the coupon rate. Think of it this way. If the price was \$1000, then in purchasing the bond, you would receive 6% every six months ($\$60/\1000) plus you would get your money back (the \$1000) at the end. The return (or YTM) must be 6% per 6 months or 12% per year compounded semi-annually—or exactly equal to the coupon rate. If you pay more than the par value (\$1000), as in this case, your return will be something less than the coupon rate. If you could buy the bond for less than the par value, say \$900, then you would earn more than 6% ($\$60/\900), and at maturity, you'd get all that you invested, \$900, plus an extra \$100. So, your return (as measured by YTM) would be something greater than the coupon rate of 6% every six months (virtually always stated as 12% per year compounded semi-annually).

So, in this case, I would choose a number less than 6% for i . If the resulting value for the right-hand side of the equation is greater than \$1225, I have not discounted the cash flows sufficiently and I must use a higher i . PVIF and PVIFA go down as i goes up. So to get a lower (higher) result, you must increase (decrease) i .

Fortunately, financial calculators and computers solve this problem very easily. For my calculator, I punch in 1000 for "FV," 60 for "pmt," 16 for "n," and -1225 for "PV." (I must use the minus sign because the cash flows are moving "opposite directions." If I buy the bond, I *pay* the \$1225 and I *receive* the interest payments and par value.) I then press "i" (yours may be "int" or you may have to push a compute, "CPT" button first). I get 4.06, which means the YTM is 8.12% per year compounded semi-annually.

D. Capital Budgeting

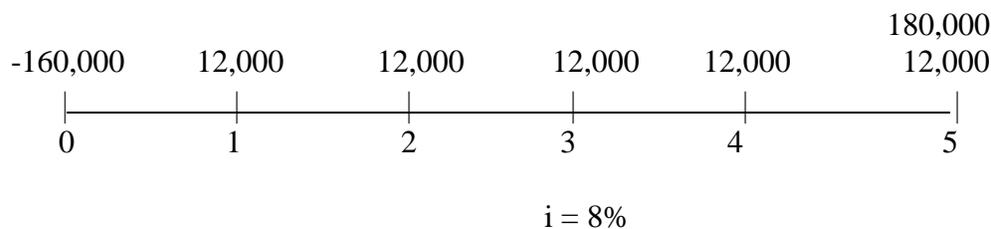
The term capital budgeting really means project evaluation. Should we introduce this new product? Should we buy a new machine or should we keep this old one? Should we build this new plant? We will discuss two methods of evaluating projects that involve TVM principles.

There are three steps to valuing anything.

- (1) Estimate the associated cash flows.
- (2) Estimate the required rate of return (or the appropriate discount rate— i).
- (3) Apply TVM principles.

For this course, we assume that the first two have been accomplished. Let's discuss two ways of accomplishing (3). These two methods are analogous to finding the price and the YTM of a bond.

Example: Say that you are considering purchasing a house for \$160,000 and renting it out. You would rent it for \$12,000 per year above expenses (tax effects included) and sell it for \$180,000 in five years. Say that you can obtain money for 8% per year. Our time line would look like this.



Net Present Value (NPV)

For this method we simply find the value at time 0 of all cash flows associated with the project.

$$\begin{aligned} \text{NPV} &= 12,000 \text{ PVIFA}(5, 8\%) + 180,000 \text{ PVIF}(5, 8\%) - 160,000 \\ &= 12,000 (3.9927) + 180,000 (.6806) - 160,000 \\ &= 47,913 + 122,508 - 160,000 \\ &= 10,421 \quad (\text{Actually, } \$10,417.50, \text{ rounding.}) \end{aligned}$$

This means, based on our estimates, the project will pay us our 8% return and we have an additional \$10,421 in value left over. In other words, if we had to pay an additional \$10,421 for the house (\$170,421), we would be indifferent between buying the house and not buying it.

Internal Rates of Return (IRR)

IRR is completely analogous to YTM for bonds. They are mathematically identical, we just call it IRR when we are talking about a project and YTM when we are talking about a bond. IRR is defined as that discount rate that sets the NPV equal to zero. So, we use the same equation, but specify NPV as zero and solve for i .

$$NPV = 0 = 12,000 PVIFA(5, i\%) + 180,000 PVIF(5, i\%) - 160,000$$

Just like with IRR, the only way we can solve this is with trial and error. All of the same comments that we made for YTM apply with IRR. My calculator tells me the IRR of this project is 9.565%. Again, the project looks good because I can obtain money for 8% and invest it for over 9½%.

There is much to capital budgeting and we have not even scratched the surface. You'll learn much more in your finance course(s).