Partial Answers to Homework #1

3.D.5 Consider again the CES utility function of Exercise 3.C.6, and assume that $\alpha_1 = \alpha_2 = 1$. Thus $u(x) = [x_1^{\rho} + x_2^{\rho}]^{1/\rho}$.

a) Compute the Walrasian demand and indirect utility functions for this utility function.

We start by restricting our attention to the case ($\rho < 1$). We maximize the equivalent utility function \([x_1^{\rho} + x_2^{\rho}]\). The first-order conditions for an interior solution are $\rho x_1^{\rho-1} = \lambda p_1$ and $\rho x_2^{\rho-1} = \lambda p_2$ where $\lambda$ is the multiplier corresponding to the budget constraint. Note that for $\rho < 1$, all solutions must be interior, otherwise $x_i^{\rho-1}$ would be undefined. (If $\rho \geq 1$, the solutions will be at the corners.)

Dividing the first-order conditions to eliminate $\lambda$ yields

$$\frac{x_1^{\rho-1}}{x_2^{\rho-1}} = \left(\frac{p_1}{p_2}\right)$$

. Thus

$$x_1 = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}} x_2$$

. Substitute this expression in the budget constraint and solve for $x_2$. Then substitute the result into the above equation to obtain $x_1$. This gives the Walrasian demand functions:

$$x_1(p, w) = \frac{w}{p_1 + \frac{1}{\rho-1} p_2^{\frac{1}{\rho-1}}} \quad x_2(p, w) = \frac{w}{p_2 + \frac{1}{\rho-1} p_1^{\frac{1}{\rho-1}}}.$$  

These can also be written

$$x_1(p, w) = \frac{p_1^{-1} w}{p_1^{-1} + p_2^{-1}} \quad x_2(p, w) = \frac{p_2^{-1} w}{p_1^{-1} + p_2^{-1}}.$$  

Substituting into the actual utility function, we find the indirect utility

$$v(p, w) = w\left[p_1^{\frac{1}{\rho-1}} + p_2^{\frac{1}{\rho-1}}\right]^{\frac{\rho}{\rho-1}}.$$  

Using the shorthand $\delta = \rho/(\rho - 1)$ we can write these as $x_i(p, w) = wp_i^{\delta-1}/[p_1^{\delta} + p_2^{\delta}]$ and $v(p, w) = w[p_1^{\delta} + p_2^{\delta}]^{-1/\delta}$. Since $\rho < 1$, $\delta < 1$.

b) Verify that these two functions satisfy all the properties of Propositions 3.D.2 and 3.D.3. For $\lambda > 0$, $x_i(\lambda p, \lambda w) = \lambda \lambda^{\delta-1} p_i^{\delta-1}/\lambda^\delta[p_1^{\delta} + p_2^{\delta}] = x_i(p, w)$, so demand is homogeneous of degree zero. Also, $p_1 x_1 + p_2 x_2 = wp_1^{\delta}/[p_1^{\delta} + p_2^{\delta}] + wp_2^{\delta}/[p_1^{\delta} + p_2^{\delta}] = w$, establishing Walras’s Law.

Concerning 3.D.3, $v(\lambda p, \lambda w) = \lambda \lambda^{-1}[p_1^{\delta} + p_2^{\delta}]^{-1/\delta} = v(p, w)$, showing that indirect utility is homogeneous of degree 0 in $(p, w)$. As the quotient of non-zero continuous functions, $v$ is continuous. Clearly $v$ is increasing in $w$ and decreasing in each $p_i$. This leaves quasi-convexity, which can be shown by considering the bordered Hessian.

c) Derive the Walrasian demand correspondence and indirect utility function for the case of linear utility and the case of Leontief utility (see Exercise 3.C.6). Show that the CES Walrasian demand and indirect utility functions approach these as $\rho$ approaches 1 and $-\infty$, respectively.
d) The elasticity of substitution between goods 1 and 2 is defined as
\[ \xi_{12}(p, w) = -\frac{\partial[x_1(p, w)/x_2(p, w)]}{\partial[p_1/p_2]} \frac{p_1/p_2}{x_1(p, w)/x_2(p, w)}. \]

Show that for the CES utility function, \( \xi_{12}(p, w) = 1/(1 - \rho) \), thus justifying its name. What is \( \xi_{12}(p, w) \) for the linear, Leontief, and Cobb-Douglas utility functions?

In part (a), we found \( x_1/x_2 = (p_1/p_2)^{1/(\rho-1)} \). Thus
\[ \theta(x_1/x_2)/\theta(p_1/p_2) = \left( \frac{p_1}{p_2} \right)^{\frac{1}{1-\rho}}. \]

It follows that \( \xi_{12} = 1/(1 - \rho) \).

3.D.6 Consider the three-good setting in which the consumer has utility function \( u(x) = (x_1 - b_1)^\alpha (x_2 - b_2)^\beta (x_3 - b_3)^\gamma \).

a) Why can you assume that \( \alpha + \beta + \gamma = 1 \) without loss of generality? Do so for the rest of the problem.

For \( \alpha + \beta + \gamma > 0 \), the transformation \( \psi(u) = u^{1/(\alpha+\beta+\gamma)} \) is increasing. Thus \( \psi(u) \) represent the same preferences and the exponents sum to 1.

b) Write down the first-order conditions for the UMP, and derive the consumer’s Walrasian demand and indirect utility functions. This system of demands is known as the linear expenditure system and is due to Stone (1954).

The utility function doesn’t make sense if \( x_1 < b_1 \), so we will focus on the interior case and assume that \( p_1b_1 + p_2b_2 + p_3b_3 < w \), which makes the interior case feasible. Ignoring corners (which can’t occur here) the Lagrangian is \( L = (x_1 - b_1)^\alpha (x_2 - b_2)^\beta (x_3 - b_3)^\gamma - \lambda(p_1x_1 + p_2x_2 + p_3x_3 - w) \).

The first-order conditions are \( \alpha u(x)/(x_1 - b_1) = \lambda p_1 \), \( \beta u(x)/(x_2 - b_2) = \lambda p_2 \), and \( \gamma u(x)/(x_3 - b_3) = \lambda p_3 \).

These reduce to \( p_1/p_2 = \alpha(x_2 - b_2)/[\beta(x_1 - b_1)] \) and \( p_1/p_3 = \alpha(x_3 - b_3)/[\gamma(x_1 - b_1)] \). Substituting in the budget constraint and solving, we find \( x_1 = b_1 + \alpha(w - p \cdot b)/p_1 \).

Then \( x_2 = b_2 + \beta(w - p \cdot b)/p_2 \) and \( x_3 = b_3 + \gamma(w - p \cdot b)/p_2 \) where \( b = (b_1, b_2, b_3) \).

The indirect utility function is then \( v(p, w) = (\alpha/p_1)^\alpha (\beta/p_2)^\beta (\gamma/p_3)^\gamma (w - p \cdot b) \).

c) Verify that these demand functions satisfy the properties listed in Propositions 3.D.2 and 3.D.3. This is straightforward.


a) Derive the Hicksian demand and expenditure functions. Check the properties listed in Propositions 3.E.2 and 3.E.3.

Solving the expenditure minimization problem leads to the first-order conditions \( p_1 = \alpha \lambda u(x)/(x_1 - b_1) \), \( p_2 = \beta \lambda u(x)/(x_2 - b_2) \), and \( p_3 = \gamma \lambda u(x)/(x_3 - b_3) \). These reduce to \( p_1/p_2 = \alpha(x_2 - b_2)/[\beta(x_1 - b_1)] \) and \( p_1/p_3 = \alpha(x_3 - b_3)/[\gamma(x_1 - b_1)] \).

Setting \( u(x) = u \), and substituting for \( (x_2 - b_2) \) and \( (x_3 - b_3) \), we find \( h_1 - b_1 = u^{\alpha - 1}\beta^\gamma \gamma p_3 - p_2^\beta p_3^\gamma \).

Thus \( h_1 = b_1 + u(p_1/\alpha)^\alpha (p_2/\beta)^\beta (p_3/\gamma)^\gamma (\alpha/p_1) \), \( h_2 = b_2 + u(p_1/\alpha)^\alpha (p_2/\beta)^\beta (p_3/\gamma)^\gamma (\beta/p_2) \), and \( h_3 = b_3 + u(p_1/\alpha)^\alpha (p_2/\beta)^\beta (p_3/\gamma)^\gamma (\gamma/p_3) \).

The expenditure function is then \( e(p, u) = p \cdot b + u(p_1/\alpha)^\alpha (p_2/\beta)^\beta (p_3/\gamma)^\gamma \).

b) Show that the derivatives of the expenditure function are the Hicksian demand function you derived in (a).

This is easily verified.
c) Verify that the Slutsky equation holds.

d) Verify that the own-substitution terms are negative and that the compensated cross-price effects are symmetric.

e) Show that $S(p, w)$ is negative semidefinite and has rank 3.