Invariants of 4-manifolds from Hopf Algebras

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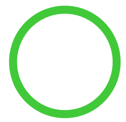
May 26, 2024

QuaSy-Con II
University of Notre Dame

Background & Motivation

Classification of (closed oriented) manifolds in low dimensions

- dim = 1: circle 
- dim = 2: surfaces 
- dim = 3: Thurston Geometrization Conjecture (Theorem by Perelman)
- dim = 4: topological: simply-connected manifolds (Freedman)
  smooth: wildly open

Construct computable invariants to distinguish (smooth) 4-manifolds.

- “classical” invariants: Euler char., signature, (co)homology, etc
- “quantum” invariants: topological quantum field theory (TQFT)
  - analytical approach: Seiberg-Witten/Donaldson-Floer
  - algebraic approach:

Algebraic construction of 4-manifold invariants (incomplete list)

- Turaev-Viro state-sum type

finite group
[Dijkgraaf-Witten, ’90]

2-group
[Yetter, ’93]

premodular category
[Crane-Yetter, ’93]

crossed braided fusion category
[C-, ’16]

$\cap$

fusion 2-category
[Douglas-Reutter, ’18]

$\cap$

universal state sum
[Walker, ’21]

- Kashaev inv based on cyclic groups
  [Kashaev, ’14]
- Dichromatic inv based on pivotal functors
  [Barenz-Barrett, ’17]
- State sums for fermionic topological order
  [Tata-Kobayashi-Bulmash-Barkeshli, ’21]
- Skein TQFTs from ribbon categories
  [Costantino-Geer-Haïoun-Patureau-Mirand, ’23]
- Trisection inv based on Hopf algebras
  [Chaidez-Cotler-Cui, ’19, ’23]

Semisimple vs non-semisimple TQFTs

- Roughly, a (d+1)-TQFT assigns to each closed d-manifold a vector space, and to each (d+1)-cobordism a linear operator, coherently (functorially).
- Semisimple (3+1)-TQFTs cannot distinguish smooth structures. [Reutter, '20]

dim = 2+1	dim = 3+1
Turaev-Viro/Barrett-Westbury TQFT from fusion categories	Turaev-Viro type TQFT from fusion 2-categories
$\Downarrow$ non-semisimple generalization	$\Downarrow$
Kuperberg invariants from (nonsemisimple) Hopf algebras	????

Goal: Construct Kuperberg-type 4-manifold invariants.

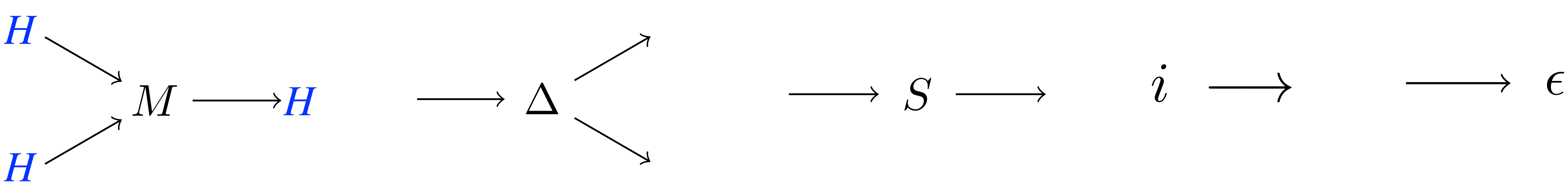
Algebraic Input: Hopf Triplets

Hopf algebra $H(M, i, \Delta, \epsilon, S)$ **(finite dimensional over $\mathbb{C}$)**

$M : H \otimes H \to H \qquad \Delta : H \to H \otimes H$

$i : \mathbb{C} \to H \qquad \epsilon : H \to \mathbb{C} \qquad S : H \to H$

Tensor diagram representations



Iterated M, Δ tensors:

$M(M \otimes I) \cdots (M \otimes I^{\otimes(n-2)}) \qquad (\Delta \otimes I^{\otimes(n-2)}) \cdots (\Delta \otimes I)\Delta$

Definition:

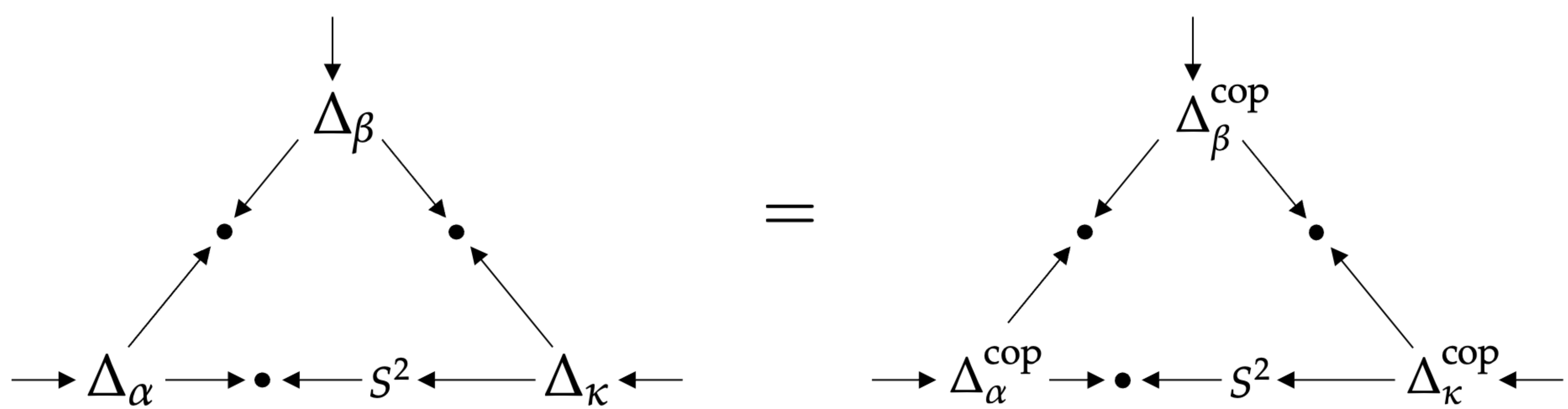
A Hopf triplet $\mathcal{H} = (H_\alpha, H_\beta, H_\gamma; \langle - \rangle)$ consists of

- three Hopf algebras $H_\alpha, H_\beta, H_\gamma$
- three pairings $\langle - \rangle_{\alpha\beta} : H_\alpha \otimes H_\beta \to \mathbb{C}$ $\quad \langle - \rangle_{\beta\gamma} : H_\beta \otimes H_\gamma \to \mathbb{C}$ $\quad \langle - \rangle_{\gamma\alpha} : H_\gamma \otimes H_\alpha \to \mathbb{C}$

tensor notation:

such that

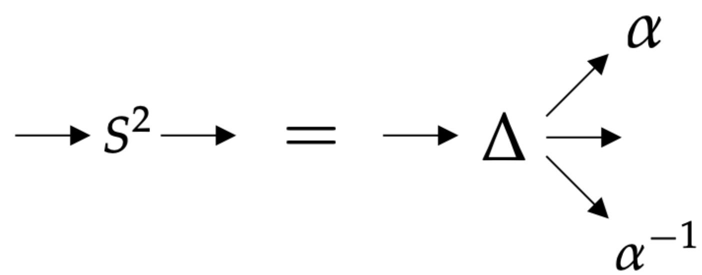
1. For $\mu\nu \in \{\alpha\beta, \beta\gamma, \gamma\alpha\}$, the induced map $\langle \cdot, \cdot \rangle_\mu^\nu : H_\mu \to H_\nu^{*,\mathrm{cop}}$ is a Hopf algebra morphism, and preserves the distinguished group-like element;
2.



Definition:

- A Hopf algebra H is balanced if

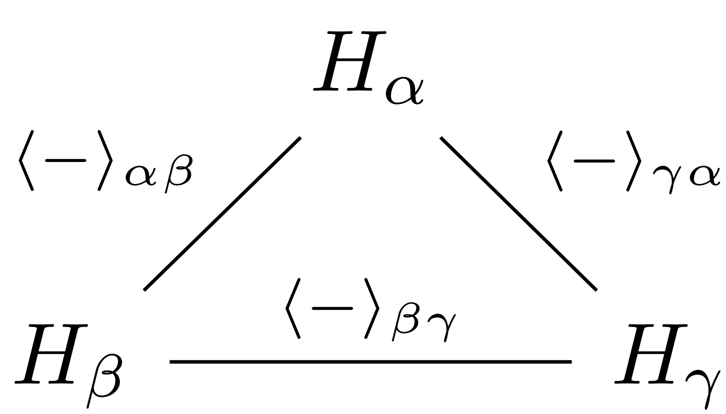
the distinguished group-like element



is involutory if $S^2 = \mathrm{Id}$

involutory $\Longleftrightarrow$ semisimple $\Longrightarrow$ balanced

- A Hopf triplet is balanced/involutory if its Hopf algebras are so.

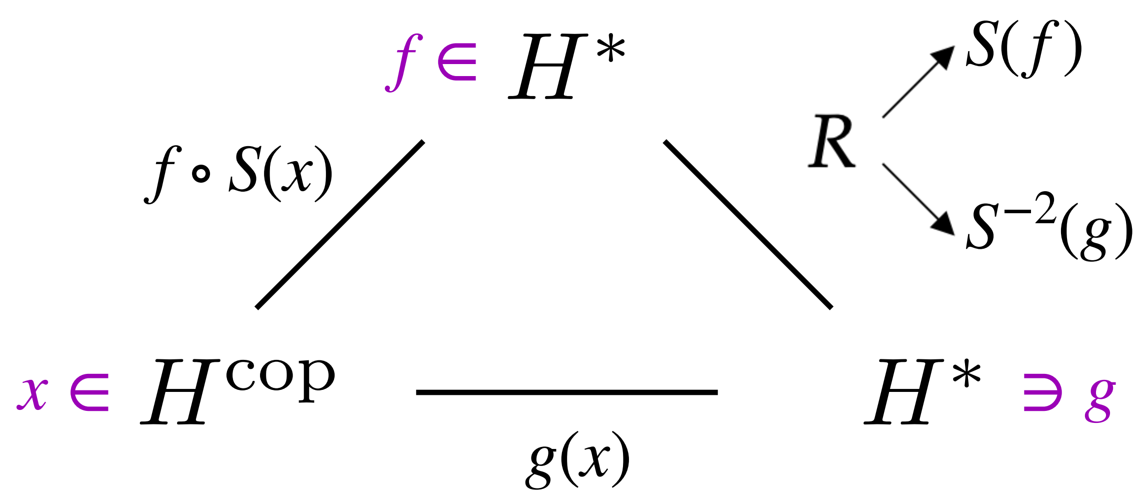


Examples of Hopf triplets

- Let (H, R) be a quasi-triangular Hopf algebra $R \in H \otimes H$

  such that $R \; (\alpha^{-1}) = a \longrightarrow$ the distinguished group-like element of *H*

  Construct a Hopf triplet $\mathscr{H}_H = (H^*, H^{\mathrm{cop}}, H^*; \langle - \rangle)$

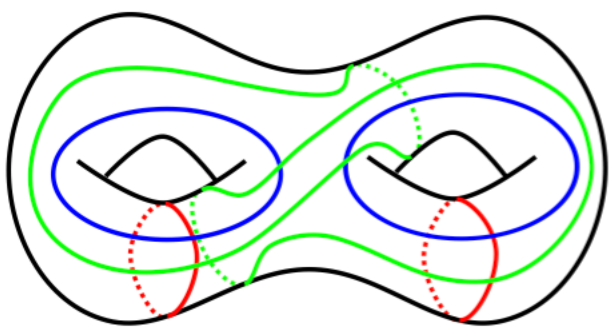


Note: this Hopf triplet is balanced/involutory if *H* is so.

The Invariant

Trisection diagrams of closed 4-manifolds

[D. Gay, R. Kirby, '12]



$(\Sigma_g, \alpha, \beta, \gamma)$

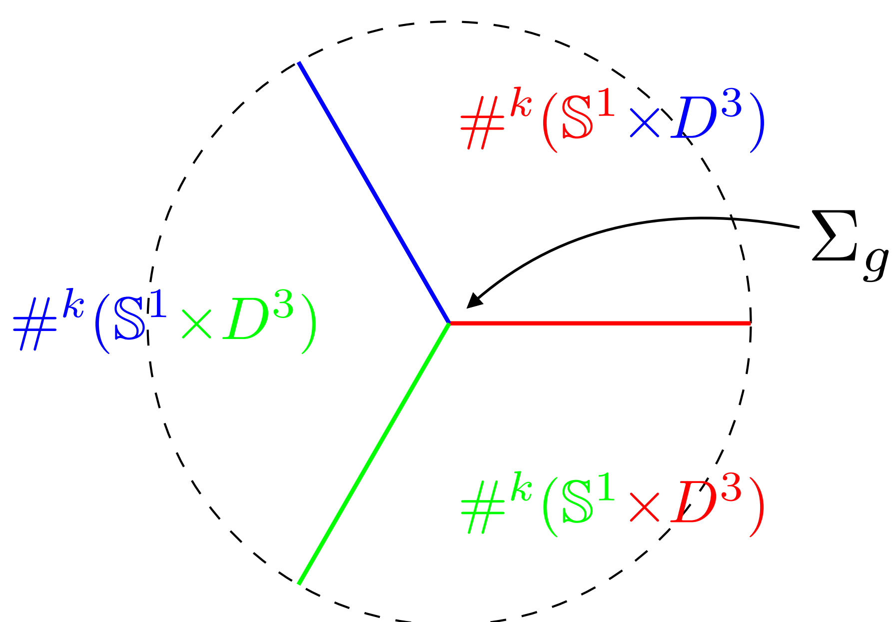
Σ_g : closed surface of genus g

$\alpha = \{\alpha_1, \cdots, \alpha_g\}$

$\beta = \{\beta_1, \cdots, \beta_g\}$

$\gamma = \{\gamma_1, \cdots, \gamma_g\}$

$(\Sigma_g, \alpha, \beta)$, $(\Sigma_g, \alpha, \gamma)$, $(\Sigma_g, \beta, \gamma)$
are Heegaard diagrams for $\#^k(\mathbb{S}^1 \times \mathbb{S}^2)$



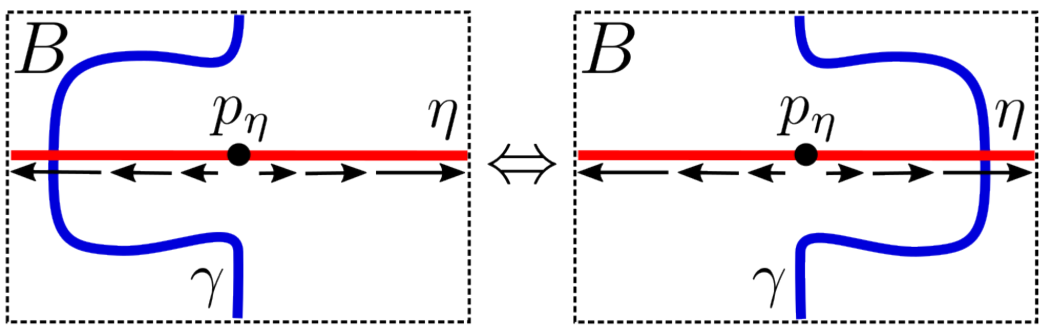
- Every closed oriented 4-manifold has trisection diagrams.
- Equivalent diagrams are related by surface isotopy, handle slides, stabilization.

Definition:

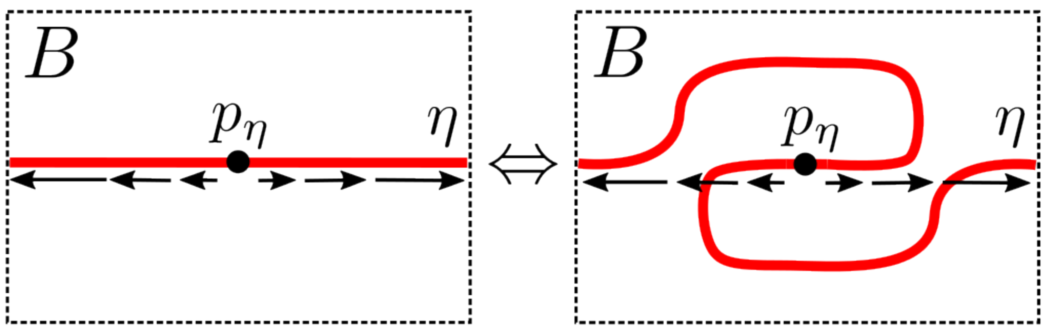
A combing on a trisection $T = (\Sigma_g, \alpha, \beta, \gamma)$ is a vector field *f* on Σ_g,

- *f* has a singularity of index -1 on each trisection curve,
- *f* has degree 0 on all components of $\partial\Sigma_g$, except the marked component.

Combing moves:



basepoint isotopy



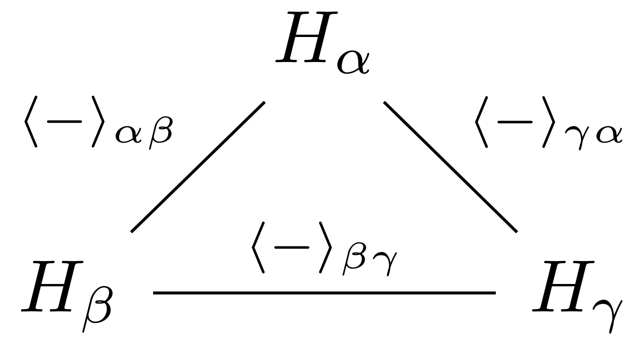
basepoint spiral

$[f]$: the set of combings equivalent to *f* under combing moves

$\mathrm{Comb}(T)$: the set of equivalence classes of combings

Theorem [Chaidez, Cotler, C-]

For a trisection diagram $T = (\Sigma_g, \alpha, \beta, \gamma)$ representing the 4-manifold X, $\mathrm{Comb}(T)$ is a torsor over $H_1(X)$.

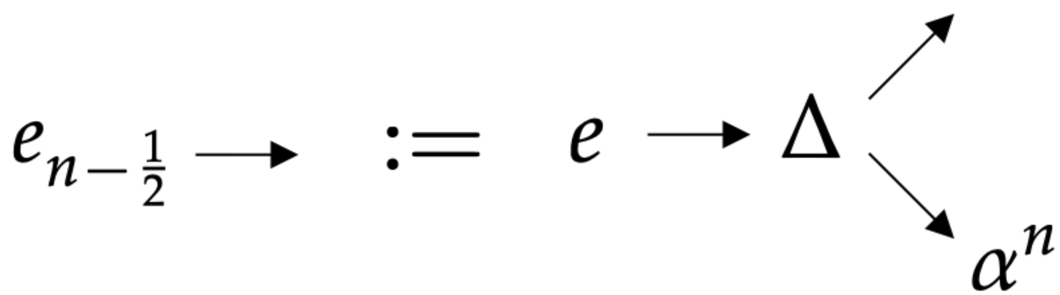


Algebraic input: a balanced Hopf triplet

$$\mathcal{H} = (H_\alpha, H_\beta, H_\gamma; \langle - \rangle)$$

$\mu \in \{\alpha, \beta, \gamma\}$

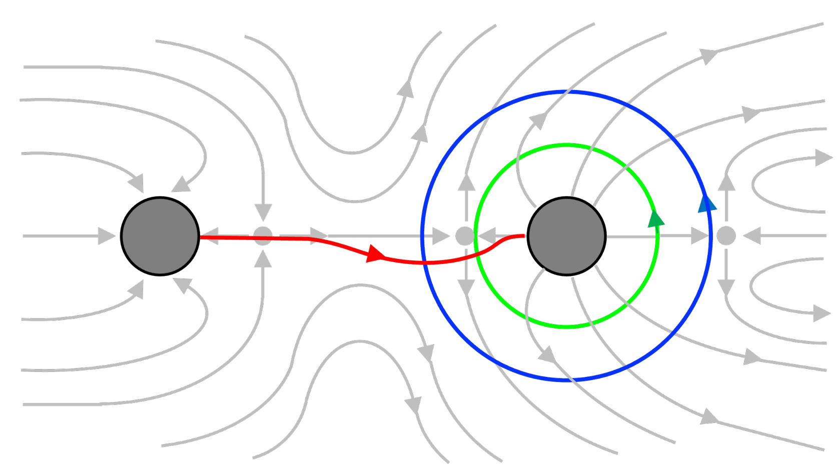
Choose a right integral $e = e^\mu \in H_\mu$, and define the generalized integral



Topological input: a closed oriented 4-manifold *M*

Choose

- a trisection diagram $T = (\Sigma_g, \alpha, \beta, \gamma)$ for M
- A combing *f* on *T*

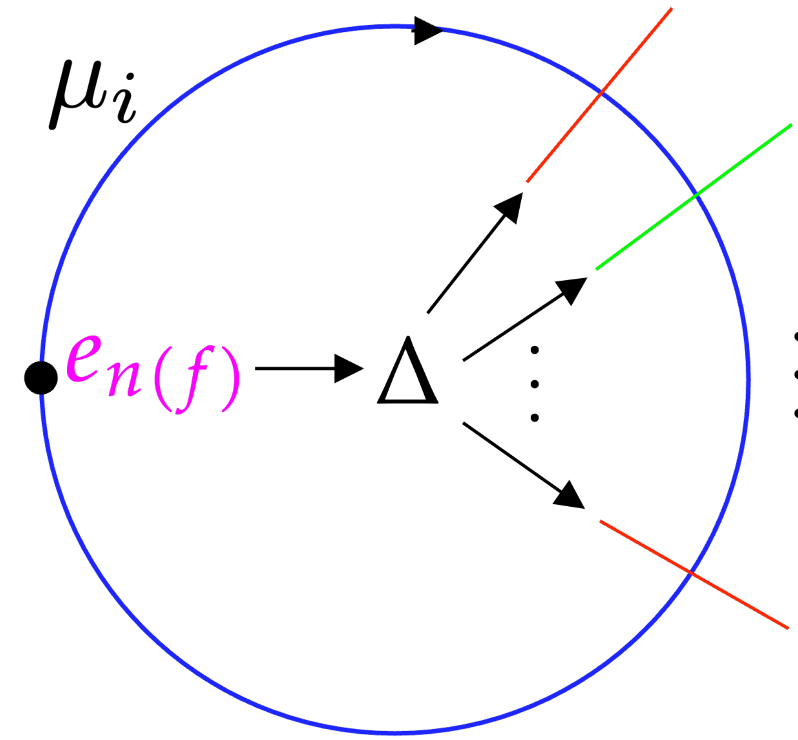


$\mu \in \{\alpha, \beta, \gamma\} \quad i = 1, \cdots, g$

To each μ_i curve, choose

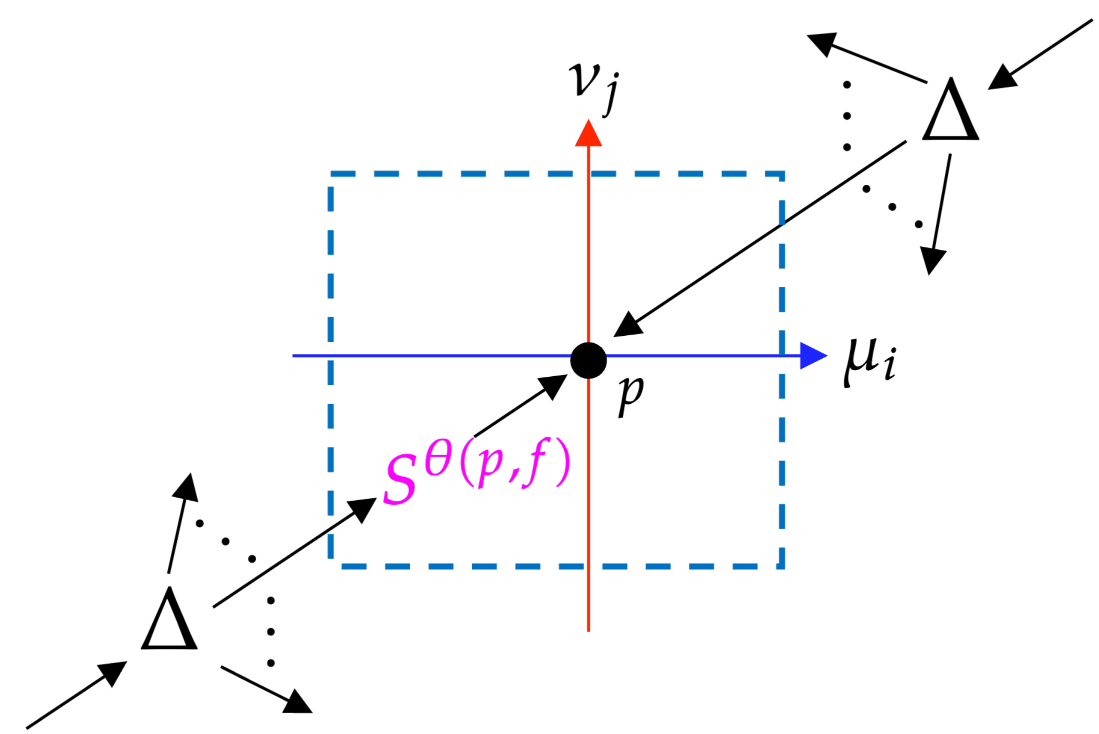
- a basepoint
- an orientation

and assign the tensor:



$\mu\nu \in \{\alpha\beta, \beta\gamma, \gamma\alpha\}$

To each crossing p between a μ_i curve and a ν_j curve, assign the tensor:



$n(f)$ and $\theta(p, f)$ are quantities depending on f

$Z(M; \mathscr{H})$:= contraction of the above tensors with certain normalization.

Involutory Case:

Theorem [Chaidez, Cotler, C-]

For a involutory Hopf triplet $\mathscr{H}$, $Z(M;\mathscr{H})$ is an invariant of the diffeomorphism class of closed oriented smooth 4-manifolds.

Theorem [Chaidez, Cotler, C-]

If H is a quasi-triangular semisimple Hopf algebra, then

$$Z(M;\mathscr{H}_H) = \mathrm{CY}(M;\mathrm{Rep}(H))$$

Crane-Yetter invariant

Theorem [Chaidez, Cotler, C-]

If $\phi : D(H) \to K$ is a Hopf algebra morphism between semisimple Hopf algebras *H* and *K*, where *D(H)* is the Drinfeld double of *H*, then

$$Z(M;\mathscr{H}(H,K,\phi)) = I(M;\mathrm{Rep}(\phi))$$

certain triplet

Dichromatic invariant

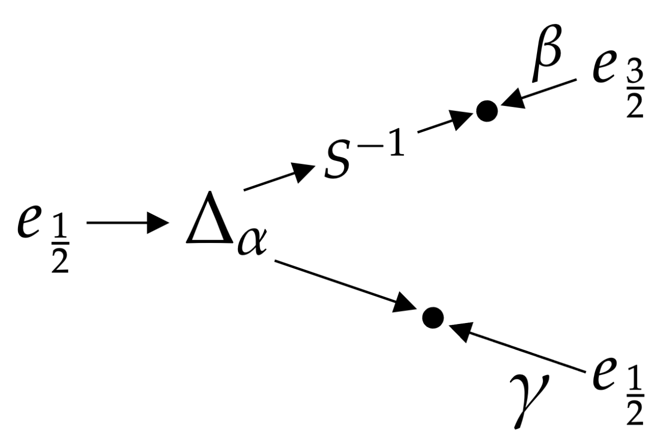
General Case:

Theorem

For a balanced Hopf triplet $\mathscr{H}$ satisfying certain normalization condition, if for 4-manifolds *M* with $H_1(M) = 0$, $Z(M; \mathscr{H})$ is an invariant of *M*.

Remark:

- For general manifolds, the invariant depends on the combing, but only on the equivalence class of the combings. Want to know what intrinsic structure of the 4-manifold elements of $\mathrm{Comb}(T)$ represent.
- The normalization condition requires the diagram below NONZERO



Known examples of triplets to date are all semisimple.

Future directions:

- Is the invariant (semisimple or not) part of a TQFT?
- Relate the invariant to other state-sum invariants besides CY.
- Find more examples of (nonsemisimple) Hopf triplets.
- Generalize to Hopf triplets in a symmetric fusion category.
- The notion of Hopf triplets should be generalized to a tri-algebra.
- Recent work of Beliakova & De Renzi on invariants 4d 2-handlebodie using unimodular ribbon Hopf algebras.

Thank You!