Math 20580
Midterm 1
February 13, 2020

Name:______________________________
Instructor:______________________________
Section:______________________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an × through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

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1. ☐ a ☐ b ☐ c ☐ d ☒ e
2. ☐ a ☒ b ☐ c ☐ d ☐ e
3. ☐ a ☒ b ☐ c ☐ d ☐ e
4. ☐ a ☐ b ☐ c ☒ d ☐ e
5. ☐ a ☐ b ☐ c ☐ d ☒ e
6. ☐ a ☐ b ☒ c ☐ d ☐ e
7. ☐ a ☒ b ☐ c ☐ d ☐ e
8. ☐ a ☐ b ☐ c ☒ d ☐ e

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Multiple Choice.

9.

10.

11.

12.

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Total.

Part I: Multiple choice questions (7 points each)

1. What can be said about the following system of linear equations?

$$\begin{cases} 2x_1 - 4x_3 = 5 \\ x_2 - 3x_3 = 3 \end{cases}$$

(a) The solution set is a subspace of $\mathbb{R}^3$

(b) The system is inconsistent

(c) There are only finitely many solutions

(d) Every solution is in $\mathbb{R}^2$

[✓ (e) none of the above]

$$\left[\begin{array}{ccc:c} \boxed{2} & 0 & -4 & 5 \\ 0 & \boxed{1} & \underset{\uparrow}{-3} & 3 \end{array}\right]$$

Consistent

free variable, so infinitely many solutions in $\mathbb{R}^3$

Not a subspace (does not contain $\begin{bmatrix}0\\0\\0\end{bmatrix} = \vec{0}$)

2. What are the values of h and k for which the matrix below is not invertible?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & k \\ -1 & h & -1 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{1} & k+1 \\ 0 & h+1 & 0 \end{bmatrix}$$

(a) $h = 0$ and $k = 0$

[✓ (b) $h = -1$ or $k = -1$]

(c) $h = 1$ and any k

(d) $h = 0$ and $k = 1$

(e) none of the above

$$\swarrow R_3 \to R_3 - (h+1) R_2$$

$$\begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{1} & (k+1) \\ 0 & 0 & \underset{\uparrow}{-(k+1)(h+1)} \end{bmatrix}$$

must be 0

So $k = -1$ or $h = -1$

3. Under which of the scenarios below does the equation $A\vec{x} = \vec{0}$ have a nontrivial solution.

   1. A is a 3×3 matrix with three pivot positions. ← pivot in every column
   2. A is a 4×4 matrix with two pivot positions. ← 2 free variables
   3. A is a 2×5 matrix with two pivot positions. ← 3 free variables
   4. A is a 5×3 matrix with three pivot positions. ← pivot in every column

   (a) 2 only [✓ (b) 2,3 only] (c) 1,4 only (d) 2,3,4 only (e) 4 only

4. If the matrices A, B are such that

$$A = \underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}}, \qquad AB = \underset{2\times 3}{\begin{bmatrix} 2 & 3 & 1 \\ 3 & 5 & 2 \end{bmatrix}},$$

then what is the matrix B?

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} -3 & 2 \\ 2 & 1 \end{bmatrix}$ [✓ (d)] $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (e) It can't be determined.

① B is 2x3 matrix, check $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$:

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 5 & 2 \end{bmatrix} \checkmark$$

② $B = A^{-1} \cdot \begin{bmatrix} 2 & 3 & 1 \\ 3 & 5 & 2 \end{bmatrix}$

$= \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 3 & 5 & 2 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

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③ Alternatively, row reduce $\left[\begin{array}{cc:ccc} 1 & 2 & 2 & 3 & 1 \\ 2 & 3 & 3 & 5 & 2 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc:ccc} 1 & 2 & 2 & 3 & 1 \\ 0 & -1 & -1 & -1 & 0 \end{array}\right] \rightarrow$

$\xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{cc:ccc} 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 \end{array}\right] \xrightarrow{R_2 \to -R_2} \left[\begin{array}{cc:c} 1 & 0 & \\ 0 & 1 & \underbrace{\begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{matrix}}_{B} \end{array}\right]$

5. Suppose that a 5×6 matrix A has a nullspace of dimension 2. How many rows of zeros does the reduced echelon form of A contain?

(a) 3 (b) 2 (c) 5 (d) 4 [✓ (e) 1]

A has $n=6$ columns

$\|$

$\text{rk}\, A + \underset{\overset{\nearrow}{2}}{\dim(\text{Nul}\, A)} \Rightarrow \text{rk}\, A = 4 \Rightarrow 4$ pivots

A has 5 rows $\Rightarrow$

$\rightarrow 5-4=1$ row of zeros.

6. Which of the following matrices has linearly independent columns?

$$A = \begin{bmatrix} 3 & -4 \\ 4 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ 2 & 0 \\ 0 & 1 \\ -2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -3 \\ 4 & -5 \\ 5 & -6 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 5 & 10 \end{bmatrix}$$

(a) A only (b) A,B only [✓ (c) A,B,C only] (d) D only (e) B, C only

2 vectors are linearly independent if not multiples of each other $\Rightarrow$ A, B, C ✓

$D \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$ D has linearly dependent columns

(or, any set of vectors containing $\vec{0}$ is linearly dependent)

7. Consider a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ of $\mathbb{R}^3$, and a linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ with the property that

$$T(\vec{b}_1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad T(\vec{b}_2) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad T(\vec{b}_3) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

If $\vec{u}$ has coordinate vector $[\vec{u}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ relative to $\mathcal{B}$, then $T(\vec{u})$ is equal to

(a) $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ [✓ (b) $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$] (c) $\vec{b}_1 - \vec{b}_2 + 2\vec{b}_3$ (d) $\begin{bmatrix} 0 \\ 6 \end{bmatrix}$ (e) $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$

$$\vec{u} = \vec{b_1} - \vec{b_2} + 2\vec{b_3}$$

$$T(\vec{u}) = T(\vec{b_1}) - T(\vec{b_2}) + 2T(\vec{b_3})$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

8. The rank of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 & 3 & 1 \\ 6 & -3 & 3 & 3 & 1 \\ 4 & -2 & 2 & 0 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1}$$

is

(a) 1 (b) 3 (c) 4 [✓ (d) 2] (e) 0

$$\longrightarrow \begin{bmatrix} 2 & -1 & 1 & 3 & 1 \\ 0 & 0 & 0 & -6 & -2 \\ 0 & 0 & 0 & -6 & -2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} \boxed{2} & -1 & 1 & 3 & 1 \\ 0 & 0 & 0 & \boxed{-6} & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\underline{\text{rank } 2}$

Part II: Partial credit questions (11 points each). Show your work.

9. (a) Find the standard matrix for each of the following transformations $\mathbb{R}^2 \to \mathbb{R}^2$:

- Counterclockwise rotation with angle $\pi/4$.
- Projection to the y-axis.

$$A = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$ has standard matrix

$$B = \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) Find the standard matrix of the linear transformation that consists of counterclockwise rotation with angle $\pi/4$, followed by projection to the y-axis.

$$\vec{x} \longrightarrow A\vec{x} \longrightarrow B(A\vec{x})$$

Matrix $BA = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

(c) Find the standard matrix of the linear transformation that consists of projection to the y-axis, followed by counterclockwise rotation with angle $\pi/4$.

$$\vec{x} \longrightarrow B\vec{x} \longrightarrow A(B\vec{x})$$

Matrix $AB = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 \end{bmatrix}$

10. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} \textcircled{1} & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|ccc} \textcircled{1} & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} \textcircled{1} & -1 & 2 & 1 & 0 & 0 \\ 0 & \textcircled{1} & 0 & -1 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|ccc} \textcircled{1} & -1 & 2 & 1 & 0 & 0 \\ 0 & \textcircled{1} & 0 & -1 & 0 & 1 \\ 0 & 0 & \textcircled{1} & 2 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_3} \left[\begin{array}{ccc|ccc} \textcircled{1} & -1 & 0 & -3 & -2 & 4 \\ 0 & \textcircled{1} & 0 & -1 & 0 & 1 \\ 0 & 0 & \textcircled{1} & 2 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|ccc} \textcircled{1} & 0 & 0 & -4 & -2 & 5 \\ 0 & \textcircled{1} & 0 & -1 & 0 & 1 \\ 0 & 0 & \textcircled{1} & 2 & 1 & -2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -4 & -2 & 5 \\ -1 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

11. Consider the matrix

$$A = \begin{bmatrix} \textcircled{1} & -3 & 0 & -1 & 0 & -2 \\ 0 & \textcircled{1} & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & \textcircled{1} & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find the rank of A, and find a basis $\mathcal{B}$ for Col(A).

rank $A = 3$

$$\mathcal{B} = \left\{ \underset{\overset{\shortparallel}{\vec{b}_1}}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}, \underset{\overset{\shortparallel}{\vec{b}_2}}{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{\overset{\shortparallel}{\vec{b}_3}}{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}} \right\}$$

pivot columns form a basis.

(b) If $\vec{a}_5$ is the fifth column of A, determine its coordinate vector $[\vec{a}_5]_{\mathcal{B}}$ relative to $\mathcal{B}$.

$$\left[\begin{array}{ccc|c} \textcircled{1} & -3 & -1 & 0 \\ 0 & \textcircled{1} & 0 & -4 \\ 0 & 0 & \textcircled{1} & 9 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{aligned} x_1 - 3x_2 - x_3 &= 0 \\ x_2 &= -4 \\ x_3 &= 9 \end{aligned} \quad \Bigg| \Rightarrow x_1 = -3$$

$$\vec{a}_5 = x_1 \vec{b}_1 + x_2 \vec{b}_2 + x_3 \vec{b}_3 \quad , \quad [\vec{a}_5]_{\mathcal{B}} = \begin{bmatrix} -3 \\ -4 \\ 9 \end{bmatrix}$$

12. Consider the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by

$$T(x_1, x_2) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 + 2x_2 \\ 2x_1 - 3x_2 \end{bmatrix}. = x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}$$

(a) Find the standard matrix of T.

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 2 \\ 2 & -3 \end{bmatrix}$$

(b) Explain why every vector $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ in the range of T satisfies $7b_1 + b_2 - 4b_3 = 0$.

$$7(x_1 - 2x_2) + (x_1 + 2x_2) - 4(2x_1 - 3x_2)$$

$$= \underline{7x_1} \underline{\underline{-14x_2}} + \underline{x_1} + \underline{\underline{2x_2}} - \underline{8x_1} + \underline{\underline{12x_2}}$$

$$= (7+1-8)\, x_1 + (-14+2+12)\, x_2$$

$$= 0$$

(c) Write down two distinct vectors $\vec{v}_1, \vec{v}_2$ that are not contained in the range of T (and make sure to explain why they are not contained in the the range).

Any vector $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ with $7b_1 + b_2 - 4b_3 \neq 0$ is NOT in the range

Examples: $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$(7 \cdot 1 + 0 - 0 = 7)$ $(7 \cdot 0 + 1 - 4 \cdot 0 = 1)$