Math 20580 Name:____________________

Midterm 2 Instructor:____________________

March 9, 2023 Section:____________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an × through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

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1. a b c d e

2. a b c d e

3. a b c d e

4. a b c d e

5. a b c d e

6. a b c d e

7. a b c d e

8. a b c d e

Multiple Choice.

9.

10.

11.

12.

Total.

Part I: Multiple choice questions (7 points each)

1. Suppose that A and B are 3×3 matrices such that $\det(A) = 3$ and $\det(B) = -2$. What is $\det(3B^T A^{-1} B)$?

   (a) -36 (b) 0 (c) 4 (d) 36 (e) none of the above

2. What are the eigenvalues of the matrix $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$?

   (a) $-4, 0$ (b) $-3, -1$ (c) $-2, 1$ (d) $1, 2$ (e) none of the above

3. The vector $\begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}$ is an eigenvector of the matrix $\begin{bmatrix} 0 & -1 & -3 \\ 3 & -1 & -2 \\ 2 & 3 & -1 \end{bmatrix}$. What is the corresponding eigenvalue?

(a) -4 (b) -2 (c) 0 (d) 2 (e) 4

4. Suppose that $T : \mathcal{P}_2 \to M_{2,2}$ is a linear transformation. Which of the following statements are always true? (Recall that $M_{2,2}$ is the vector space of 2×2 matrices, and $\mathcal{P}_2$ is the vector space of polynomials of degree at most 2. Also recall that $\text{rank}(T)$ is the dimension of the range of T, and $\text{nullity}(T)$ is the dimension of the kernel of T.)

I. $\text{rank}(T) + \text{nullity}(T) = 4$.

II. T is one-to-one if and only if $\text{nullity}(T) = 0$.

III. The range of T is a subspace of $\mathcal{P}_2$.

(a) I only (b) II only (c) I, III only (d) II, III only (e) none of them

5. Recall that $M_{2,2}$ is the vector space of 2×2 matrices. Consider the linear transformation

$$T : M_{2,2} \to M_{2,2}, \qquad T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a + 2b + c & b - c + d \\ -a - 3c + 2d & a + 3b + d \end{bmatrix}.$$

Which of the following vectors is in the kernel of T?

(a) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & -1 \\ -1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} -3 & 1 \\ 1 & -2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$

6. Which of the following statements are always true for an $n \times n$ matrix A?

I. If A is invertible, then 0 is not an eigenvalue of A.

II. If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable.

III. Every matrix similar to A has the same characteristic polynomial as A.

(a) I only (b) I, II only (c) I, III only (d) II, III only (e) I, II, III

7. Consider the linear system

$$\begin{cases} x_1 + x_2 = 3, \\ x_1 - x_2 = 2. \end{cases}$$

According to Cramer's rule, what is x_2?

(a) $\dfrac{\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}}$ (b) $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$ (c) $\dfrac{\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}}$ (d) $\dfrac{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}}$ (e) $\dfrac{\begin{vmatrix} 3 & 3 \\ 2 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}}$

8. Recall that $M_{2,2}$ is the vector space of 2×2 matrices. Consider the function

$$T : M_{2,2} \to \mathbb{R}^2, \qquad T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a^2 + b^2 \\ c^2 + d^2 \end{bmatrix}.$$

Which of the following statements are true?

I. T is a linear transformation.

II. T is not a linear transformation because $T(\vec{0}) \neq \vec{0}$.

III. T is not a linear transformation because there exist A in $M_{2,2}$ and a scalar k such that $T(kA) \neq kT(A)$.

(a) none of them (b) I only (c) II only (d) III only (e) II, III only

Part II: Partial credit questions (11 points each). Show your work.

9. Consider the bases

$$\mathcal{B} = \left\{1 - x, x - x^2, x^2\right\} \quad \text{and} \quad \mathcal{C} = \{1 - x + x^2, 1 + 3x, 2 - x - 2x^2\}$$

of $\mathcal{P}_2$ (the vector space of polynomials of degree at most 2 in the variable x).

(a) Find the change-of-basis matrix $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ from $\mathcal{C}$ to $\mathcal{B}$.

(b) Suppose that $p(x)$ is a vector in $\mathcal{P}_2$ with $[p(x)]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. What is $[p(x)]_{\mathcal{B}}$?

10. Recall that $\mathcal{P}_2$ is the vector space of polynomials of degree at most 2 in the variable x. Consider the linear transformation

$$T : \mathcal{P}_2 \to \mathcal{P}_2, \quad T(p(x)) = p(x) - (1+x)p'(x),$$

where $p'(x)$ is the derivative of $p(x)$.

(a) Verify that T can be expressed more explicitly as

$$T(a + bx + cx^2) = (a - b) - 2cx - cx^2.$$

(b) Let $\mathcal{E} = \{1, x, x^2\}$ be the standard basis of $\mathcal{P}_2$. Find the matrix $[T]_{\mathcal{E}} = \underset{\mathcal{E}\leftarrow\mathcal{E}}{[T]}$ of T with respect to $\mathcal{E}$.

(c) Find a basis for the kernel of T and a basis for the range of T.

11. Consider the matrix

$$A = \begin{bmatrix} 1 & t & -1 \\ 0 & 3 & t \\ 2 & 1 & -2 \end{bmatrix},$$

where t is some real number.

(a) Calculate the determinant of A. (Your answer may depend on t.)

(b) Find all values of t such that A is invertible.

12. Let A be the matrix

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 2 & -1 & 4 \\ -1 & 1 & -1 \end{bmatrix}.$$

The characteristic polynomial of A is $\det(A - \lambda I) = (1-\lambda)^2(-2-\lambda)$.

(a) What are the eigenvalues of A?

(b) Diagonalize A, that is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.