Math 20580 Name: Solutions

Practice Midterm 2 Instructor:

March 5, 2015 Section:

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an × through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

1. a b c d ~~e~~
2. a b c ~~d~~ e
3. a ~~b~~ c d e
4. a ~~b~~ c d e
5. ~~a~~ b c d e
6. a b ~~c~~ d e
7. a ~~b~~ c d e
8. a b c ~~d~~ e

Multiple Choice.

9.

10.

11.

12.

Total.

Part I: Multiple choice questions (7 points each)

1. Let V be the vector space of all functions $f(x)$ where $f : \mathbb{R} \to \mathbb{R}$.
   Which of the following are subspaces of V?

   A. the constant functions;

   B. functions with $\lim_{x\to\infty} f(x) = 3$;

   C. functions with $f(1) = 1$;

   D. functions with $f(0) = 0$.

   (a) A, B, C and D (b) A, B and C only (c) B, C and D only
   (d) B and D only (e) A and D only.

B & C are not subspaces.

For example, if $\lim_{x\to\infty} f(x) = 3$ then

$$\lim_{x\to\infty} 2f(x) = 6.$$

2. Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most 2, and let $\mathcal{B}$ be the basis

$$\mathcal{B} = \{1, 1+x, 1+x^2\}.$$

Find the $\mathcal{B}$-coordinates $[p]_{\mathcal{B}}$ of the polynomial $p(x) = (1-x)^2$.

(a) $\begin{bmatrix}1\\2\\3\end{bmatrix}$ (b) $\begin{bmatrix}3\\2\\1\end{bmatrix}$ (c) $\begin{bmatrix}2\\3\\-1\end{bmatrix}$ (d) $\begin{bmatrix}2\\-2\\1\end{bmatrix}$ (e) $\begin{bmatrix}1\\1\\-1\end{bmatrix}$

$$p(x) = 1 - 2x + x^2$$
$$= 2.(1) - 2.(1+x) + (1+x^2)$$

3. Let H be a subspace of a vector space V, and suppose that V has dimension d. Which of the following statements are true?

A. $\dim(H) \leq \dim(V)$; ✓

B. a linearly independent set of vectors in H is also linearly independent in V; ✓

C. d vectors which span V will be linearly independent; ✓

D. d vectors which span H will also span V. ✗

(a) A, B, C and D (b) A, B and C only (c) B, C and D only
(d) B and D only (e) A and D only.

Only D. is false. Spanning H does not mean vectors span all of V.

4. A linear transformation

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

has outputs $T\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$ and $T\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$.

Find $T\begin{bmatrix}1\\1\end{bmatrix}$.

(a) $\begin{bmatrix}1\\1\end{bmatrix}$ (b) $\begin{bmatrix}4/5\\1\end{bmatrix}$ (c) $\begin{bmatrix}3\\1\end{bmatrix}$ (d) $\begin{bmatrix}4\\1\end{bmatrix}$ (e) $\begin{bmatrix}4\\5\end{bmatrix}$

Let $\mathcal{B} = \left\{ \begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}-1\\3\end{pmatrix} \right\}$ be a basis of $\mathbb{R}^2$

$$\left[\begin{pmatrix}1\\1\end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix}2/5\\1/5\end{pmatrix}.$$

Therefore $T\begin{pmatrix}1\\1\end{pmatrix} = T\left(\frac{2}{5}\begin{pmatrix}3\\1\end{pmatrix} + \frac{1}{5}\begin{pmatrix}-1\\3\end{pmatrix}\right) = \frac{2}{5}\begin{pmatrix}1\\2\end{pmatrix} + \frac{1}{5}\begin{pmatrix}2\\1\end{pmatrix}$

5. The vector $\begin{bmatrix}1\\0\\1\end{bmatrix}$ is an eigenvector for the matrix

$$\begin{bmatrix}2 & 5 & 1\\1 & 7 & -1\\1 & 0 & 2\end{bmatrix}$$

What is the corresponding eigenvalue?

(a) 3 (b) 1 (c) 0 (d) -1 (e) 2

$$\begin{pmatrix}2 & 5 & 1\\1 & 7 & -1\\1 & 0 & 2\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}3\\0\\3\end{pmatrix}$$

so the eigenvalue is 3.

6. What are the eigenvalues of the matrix $\begin{bmatrix}5 & 4\\-2 & -1\end{bmatrix}$?

(a) $0, 1$ (b) $5, -1$ (c) $1, 3$ (d) $0, 2$ (e) $-1, -3$

$$\begin{aligned}\det(A-\lambda I) &= (5-\lambda)(-1-\lambda)+8\\ &= -5-4\lambda+\lambda^2+8\\ &= \lambda^2-4\lambda+3\\ &= (\lambda-3)(\lambda-1)\end{aligned}$$

So $\lambda = 1, 3$

7. Suppose A is a 3×3 matrix, that has $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ as an eigenvector with eigenvalue 2, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as an eigenvector with eigenvalue -1. Compute $A^3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(a) $\begin{bmatrix} 8 \\ 16 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 8 \\ 16 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

(e) cannot be computed from the given information

$$A^3\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = A^3\left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = A^3\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + A^3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= 8\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ -1 \end{pmatrix}$$

8. Let P_3 be the vector space of polynomials of degree at most 3. Find the dimension of the subspace of P_3 spanned by $1 + x^2$, $x + 2x^2 + x^3$ and $1 + x + x^3$.

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

The coordinates of the polynomials relative to the basis $\mathcal{B} = \{1, x, x^2, x^3\}$ are

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$ respectively.

The matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

There are pivots in each column so the polynomials are linearly independent.

Part II: Partial credit questions (11 points each). Show your work.

9. Find a basis for the Row space, Row(A), of the matrix

$$A = \begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 0 & 2 & 3 \\ 1 & 1 & -5 & 6 \\ 1 & -1 & -1 & 2 \end{bmatrix}.$$

$$A \rightsquigarrow \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & -7 & 3 \\ 0 & -1 & -3 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -10 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The non-zero rows form a basis, namely

$$\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -10 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

10. Let $\mathcal{B} = \{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\}$ and $\mathcal{C} = \{\begin{bmatrix}2\\-1\end{bmatrix}, \begin{bmatrix}-1\\0\end{bmatrix}\}$ be two bases of $\mathbb{R}^2$.

Find the change of coordinate matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{\mathcal{P}}$ sending a $\mathcal{B}$ coordinate vector $[\vec{x}]_{\mathcal{B}}$ to the $\mathcal{C}$ coordinate vector $[\vec{x}]_{\mathcal{C}}$.

Let $B = \begin{pmatrix}1 & 3\\2 & 4\end{pmatrix}$, $C = \begin{pmatrix}2 & -1\\-1 & 0\end{pmatrix}$

Then $\underset{C\leftarrow B}{P} = C^{-1}B = \frac{1}{(-1)}\begin{pmatrix}0 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}1 & 3\\2 & 4\end{pmatrix}$

$$= -\begin{pmatrix}2 & 4\\5 & 11\end{pmatrix}$$

$$= \begin{pmatrix}-2 & -4\\-5 & -11\end{pmatrix}$$

11. Is the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix}$ diagonalizable? If so, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If A is not diagonalizable, explain why not.

(Hint: The eigenvalues of A are 1 and 3.)

Eigenspace for $\lambda = 1$ are solns. to $(A-I)x = 0$

$$A - I = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

ie. $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ & basis is $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Eigenspace for $\lambda = 3$ are solns. to $(A-3I)x = 0$

$$A - 3I = \begin{pmatrix} -1 & 1 & -1 \\ 0 & -2 & 0 \\ -1 & -1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 & -1 \\ 0 & -2 & 0 \\ 0 & -2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

ie. $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ & basis is $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Take $P = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ & $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

12. Let P_3 be the vector space of polynomials of degree at most 3 and P_2 be the space of polynomials of degree at most 2.

Consider the linear transformation

$$T : P_3 \to P_2$$

given by $T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2$.

(a) Write down bases $\mathcal{B}_3$ and $\mathcal{B}_2$ of P_3 and P_2 respectively.

$\mathcal{B}_3 = \{1, x, x^2, x^3\}$

$\mathcal{B}_2 = \{1, x, x^2\}$

(b) Find the matrix of T relative to the bases $\mathcal{B}_3$ and $\mathcal{B}_2$.

$$\left.\begin{aligned} T(1) &= 0 \\ T(x) &= 1 \\ T(x^2) &= 2x \\ T(x^3) &= 3x^2 \end{aligned}\right\}$$

The matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$