Part I: Multiple choice questions (7 points each)

1. Which of the following functions is a solution of the initial value problem

$$(y' - \sin x)^2 = 1 + x^2 - y^2, \quad y(0) = 1$$

(a) ~~$-\sin x$~~ (b) $x \sin x + \cos x$ (c) $\cos x$ (d) ~~$x \cos x - \sin x$~~ (e) ~~$\sin x - \cos x$~~

$y(0)=0$ $y(0)=1$ ✓ $y(0)=1$ ✓ $y(0)=0$ $y(0)=-1$

$\boxed{y' = x\cos x + \sin x - \sin x = x\cos x}$

$\boxed{y' = -\sin x} \longrightarrow (y' - \sin x)^2 = 4\sin^2 x$

$1 + x^2 - y^2 = 1 + x^2 - \cos^2 x$

different!!

$(y' - \sin x)^2 = (x\cos x - \sin x)^2 = x^2\cos^2 x - 2x\cos x \sin x + \sin^2 x$

$1 + x^2 - y^2 = 1 + x^2 - (x \sin x + \cos x)^2 = 1 + \underbrace{x^2 - x^2\sin^2 x}_{x^2\cos^2 x} - 2x\sin x\cos x - \overbrace{\cos^2 x}^{\sin^2 x}$

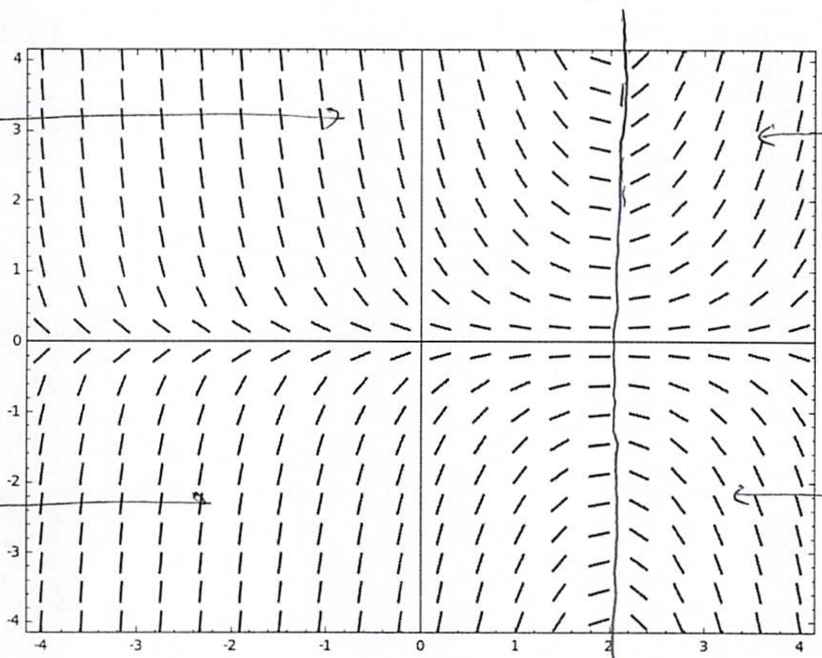
$= x^2\cos^2 x - 2x\sin x\cos x + \sin^2 x$

Same!

2. Determine $f(t, y)$ if the differential equation $y' = f(t, y)$ has direction field (the value of t is measured on the horizontal axis, and the value of y on the vertical axis)

negative

positive



positive

negative

(a) $\sin(t) + y$ (b) $y + t^2$ (c) $t \sin(y)$ (d) $ty - 2y$ (e) $e^y(t - 1)$

$t=0$, $y>0$ would be positive

$t=1$, $y<-1$ would be negative

$y = \pm\pi$ would be 0

$t=1$ would be 0 for all y

3. Consider the orthogonal vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$ and let $V = \text{Span}\{\vec{v}_1, \vec{v}_2\}$. The matrix of the projection onto V is

(a) $\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$ (b) $\begin{bmatrix} 1/9 & -2/9 & 0 \\ 2/9 & 2/9 & 0 \\ 2/9 & -1/9 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -2 \\ 2 & 2 \\ 2 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 5/9 & -2/9 & 4/9 \\ -2/9 & 8/9 & 2/9 \\ 4/9 & 2/9 & 5/9 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 2 \\ -2 & 2 & 1 \end{bmatrix}$

$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \cdot \vec{v}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$ $\quad \vec{u}_2 = \frac{1}{\|\vec{v}_2\|} \cdot \vec{v}_2 = \begin{bmatrix} -2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$ $\quad U = [\vec{u}_1 \ \vec{u}_2]$

$$U \cdot U^T = \begin{bmatrix} 1/3 & -2/3 \\ 2/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & 2/3 & -1/3 \end{bmatrix} = \begin{bmatrix} 5/9 & -2/9 & 4/9 \\ -2/9 & 8/9 & 2/9 \\ 4/9 & 2/9 & 5/9 \end{bmatrix}$$

4. Let A be an $m \times n$ matrix with linearly independent columns and let $\vec{b}$ in $\mathbb{R}^m$ be a vector which is not in $Col(A)$. Which of the following statements may be false?

(a) There exists a vector $\vec{x}$ in $\mathbb{R}^n$ with $A\vec{x} - \vec{b}$ perpendicular to $Col(A)$.

(b) $\det(A^T A) \neq 0$.

(c) $m > n$.

(d) The vector $\vec{b}$ is not the zero vector.

(e) $\det(AA^T) \neq 0$.

$\to$ $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$A \cdot A^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\det(AA^T) = 0$

5. Find the solution to the initial value problem

$$t\frac{dy}{dt} + 3y = \frac{t}{1+t^4}, \qquad y(1) = 0.$$

(a) $y = \ln\left(\frac{1+t^4}{2t^3}\right)$ (b) $y = t^3 - 1$ (c) $y = \frac{1}{4t^3} \cdot \ln\left(\frac{1+t^4}{2}\right)$

(d) $y = \frac{1}{2} \cdot \arctan(t^2) - \pi/8$ (e) $y = \frac{4t^3 - 4}{1+t^4}$

$$\frac{dy}{dt} + \underbrace{\frac{3}{t}}_{p(t)} y = \underbrace{\frac{1}{1+t^4}}_{g(t)}$$

$$\int \frac{t^3}{1+t^4}\,dt = \int \frac{du}{4u}$$

$u = 1+t^4$

$du = 4t^3 dt$

$$= \frac{\ln u}{4} = \frac{\ln(1+t^4)}{4}$$

$$\mu(t) = e^{\int 3/t\,dt} = e^{3\ln t} = t^3$$

$$y(t) = \frac{\int \frac{t^3}{1+t^4}dt}{t^3} = \frac{\frac{1}{4}\ln(1+t^4) + c}{t^3} = \frac{\ln(1+t^4) - \ln 2}{4t^3} = \frac{1}{4t^3} \cdot \ln\left(\frac{1+t^4}{2}\right)$$

$t = 1, y = 0 \qquad c = -\frac{\ln 2}{4}$

6. Which of the following functions can be used as an integrating factor for the equation $y' + ty = \cos t$?

(a) t (b) $e^{t^2/2}$ (c) $t^2/2$ (d) e^t (e) $e^{\cos t}$

$$y' + \underbrace{t}_{p(t)} y = \underbrace{\cos t}_{g(t)}$$

$$\mu(t) = e^{\int p(t)dt} = e^{t^2/2}$$

7. The ordinary differential equation

$$(2xy^2 + 2y) + (2x^2y + 2x)y' = 0$$

is

(a) linear (b) autonomous (c) separable (d) an equation of order 2
(e) none of the above.

$$2y(xy+1) + 2x(xy+1)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2y(xy+1)}{2x(xy+1)} = \frac{-y}{x}$$

$$\frac{dy}{y} = -\frac{dx}{x}$$ separable

8. The solution of the initial value problem

$$x \cdot \frac{dy}{dx} = y + xy, \qquad y(1) = 2$$

is the function

(a) $y = \dfrac{e^x}{2(x+1)}$ (b) $y = \ln(x) + 2x$ (c) $y = x^2 + x$ (d) $y = 2$ (e) $y = 2xe^{x-1}$.

$$x\frac{dy}{dx} = y(x+1)$$

$$\int \frac{dy}{y} = \int \frac{x+1}{x}\,dx$$

$$\ln y = \int\left(1+\frac{1}{x}\right)dx = x + \ln x + C$$

$x=1$
$y=2$

$$\ln 2 = 1 + 0 + C$$

$$\boxed{C = \ln 2 - 1}$$

$$y = e^{x+\ln x+\ln 2-1}$$

$$= e^x \cdot e^{\ln x} \cdot e^{\ln 2} \cdot e^{-1}$$

$$= e^x \cdot x \cdot 2 \cdot e^{-1}$$

$$= 2x e^{x-1}$$

Part II: Partial credit questions (11 points each). Show your work.

9. Let $W = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ -4 \\ 4 \\ -2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -3 \end{bmatrix}.$$

(a) Apply the Gram-Schmidt process to find an orthonormal basis for W.

$$\vec{w}_1 = \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \qquad \vec{v}_2 \cdot \vec{w}_1 = -4 \qquad \vec{v}_3 \cdot \vec{w}_1 = 4$$

$$\vec{w}_1 \cdot \vec{w}_1 = 4$$

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1} \cdot \vec{w}_1 = \begin{bmatrix} 2 \\ -4 \\ 4 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 3 \\ -3 \end{bmatrix} \qquad \vec{v}_3 \cdot \vec{w}_2 = 0$$

$$\vec{w}_2 \cdot \vec{w}_2 = 36$$

$$\vec{w}_3 = \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1 - \frac{\vec{v}_3 \cdot \vec{w}_2}{\vec{w}_2 \cdot \vec{w}_2} \cdot \vec{w}_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ -2 \end{bmatrix} \qquad \vec{w}_3 \cdot \vec{w}_3 = 16$$

$\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is orthogonal, $\|\vec{w}_1\| = 2$, $\|\vec{w}_2\| = 6$, $\|\vec{w}_3\| = 4$

divide by length

$$\boxed{\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} 3/6 \\ -3/6 \\ 3/6 \\ -3/6 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \quad \vec{u}_3 = \begin{bmatrix} -2/4 \\ 2/4 \\ 2/4 \\ -2/4 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}}$$

(b) Find the QR decomposition of the matrix A with columns $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

$$R = \begin{bmatrix} \|\vec{w}_1\| & \frac{\vec{v}_2 \cdot \vec{w}_1}{\|\vec{w}_1\|} & \frac{\vec{v}_3 \cdot \vec{w}_1}{\|\vec{w}_1\|} \\ 0 & \|\vec{w}_2\| & \frac{\vec{v}_3 \cdot \vec{w}_2}{\|\vec{w}_2\|} \\ 0 & 0 & \|\vec{w}_3\| \end{bmatrix} = \begin{bmatrix} 2 & \frac{-4}{2} & \frac{4}{2} \\ 0 & 6 & \frac{0}{6} \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 2 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

<u>Alternatively</u>, $R = Q^T A = \begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & -4 & 3 \\ -1 & 4 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 2 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

10. Let $A = \begin{bmatrix} -1 & 1 \\ -6 & 4 \\ 2 & -1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$.

(a) Find the least squares solution to the equation $A\vec{x} = \vec{b}$.

$$A^TA = \begin{bmatrix} 41 & -27 \\ -27 & 18 \end{bmatrix} \qquad A^T\vec{b} = \begin{bmatrix} -13 \\ 9 \end{bmatrix}$$

Solve $A^TA\vec{x} = A^T\vec{b}$

$$\left[\begin{array}{cc:c} 41 & -27 & -13 \\ -27 & 18 & 9 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{9}R_2} \left[\begin{array}{cc:c} 41 & -27 & -13 \\ -3 & 2 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 + 13R_2} \left[\begin{array}{cc:c} 2 & -1 & 0 \\ -3 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{cc:c} -1 & 1 & 1 \\ -3 & 2 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{cc:c} -1 & 1 & 1 \\ 0 & -1 & -2 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{cc:c} -1 & 0 & -1 \\ 0 & -1 & -2 \end{array}\right]$$

$$\xrightarrow[R_2 \to (-1)R_2]{R_1 \to (-1)R_1} \left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \qquad \boxed{\hat{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}} \text{ is least squares solution.}$$

(b) Find the vector in the column space of A which is closest to $\vec{b}$.

$$\hat{b} = A \cdot \hat{x} = \begin{bmatrix} -1 & 1 \\ -6 & 4 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

11. A tank initially contains 50 liters of water and 20 grams of salt. Water containing a salt concentration of 2 g/L enters the tank at the rate of 5 L/min, and the well-stirred mixture leaves the tank *at the same rate.*

(a) Find an expression for the $\underbrace{\text{amount of salt in the tank at any time}}_{y(t)}$ t.

$$\begin{cases} \dfrac{dy}{dt} = \underset{g/L}{2} \cdot \underset{L/min}{5} - \underset{g/L}{\dfrac{y}{50}} \cdot \underset{L/min}{5} \\ \underset{g}{y(0) = 20} \end{cases}$$

$$\frac{dy}{dt} = 10 - \frac{y}{10} \Rightarrow y(t) = 100 + c \cdot e^{-t/10} = \boxed{100 - 80\, e^{-t/10}}$$

$$t=0: \quad 20 = 100 + C \Rightarrow C = -80$$

(b) How long does it take for the amount of salt to reach 60 grams.

$$y(t) = 60$$

$$100 - 80\, e^{-t/10} = 60$$

$$40 = 80 e^{-t/10}$$

$$\frac{1}{2} = e^{-t/10}$$

$$\ln\left(\frac{1}{2}\right) = -\frac{t}{10}$$

$$t = -10 \ln\left(\frac{1}{2}\right)$$

$$\boxed{t = 10 \ln 2}$$

(c) Find the approximate amount of salt after 100 years.

100 years = t minutes where $t \approx \infty$

$$\lim_{t \to \infty} y(t) = 100$$

So approximately

$\boxed{\text{100 g of salt after 100 years}}$

12. (a) Find, in terms of y_0, the solution of the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = t \cdot e^y \\ y(0) = y_0 \end{cases}$$

$$\int \frac{dy}{e^y} = \int t\,dt$$

$$-e^{-y} = t^2/2 + C$$

$t=0$, $y=y_0$ $\Rightarrow$ $\boxed{-e^{-y_0} = C}$

$\Rightarrow$ $e^{-y} = e^{-y_0} - t^2/2$

$$-y = \ln\left(e^{-y_0} - t^2/2\right)$$

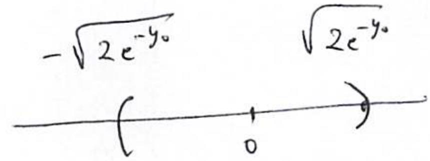
$$\boxed{y = -\ln\left(e^{-y_0} - t^2/2\right)}$$

(b) Find the maximal interval on which the solution to the initial value problem above exists, and explain how this interval depends on y_0.

We need $e^{-y_0} - \frac{t^2}{2} > 0$

$\iff t^2 < 2e^{-y_0}$

$\iff -\sqrt{2e^{-y_0}} < t < \sqrt{2e^{-y_0}}$



interval $I = \left(-\sqrt{2e^{-y_0}}, \sqrt{2e^{-y_0}}\right)$

The bigger y_0, the smaller the interval!