Math 20580 Name: Solutions

Midterm 3 Instructor:

April 16, 2015 Section:

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an × through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

1. a b c ~~d~~ e
2. a b c ~~d~~ e
3. a ~~b~~ c d e
4. a b c ~~d~~ e
5. a b c ~~d~~ e
6. a b c d ~~e~~
7. a ~~b~~ c d e
8. a b c d ~~e~~

Multiple Choice.

9.

10.

11.

12.

Total.

Part I: Multiple choice questions (7 points each)

1. The closest point to $\begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$ in the subspace of $\mathbb{R}^3$ spanned by $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ is

(a) $\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ 1/2 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$ (d) $\begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}$ (e) $\begin{bmatrix} 9 \\ 12 \\ 0 \end{bmatrix}$

This is $\text{proj}_{\text{Span}\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}\right\}} \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$.

As the spanning vectors are orthogonal, this is

$$\frac{\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \frac{\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}}{24} \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$

2. Consider the vector $\vec{w} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and the line $W = \text{Span}\{\vec{w}\}$ spanned by $\vec{w}$. $A\vec{x}$ computes the projection of a vector $\vec{x}$ onto W if

(a) $A = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$ (b) $A = \begin{bmatrix} 9/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix}$ (c) $A = \begin{bmatrix} 1/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1/10 & -3/10 \\ -3/10 & 9/10 \end{bmatrix}$ (e) $A = \begin{bmatrix} 1/10 & -3 \\ -3/10 & 9 \end{bmatrix}$

$\text{proj}_W \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, $\text{proj}_W \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{-3}{10} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

3. The matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 4 \\ 1 & -1 \\ 1 & 4 \end{bmatrix}$ factors as $A = QR$ where $Q = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$ and

(a) $R = \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix}$ (b) $R = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$ (c) $R = \begin{bmatrix} 4 & 6 \\ 6 & 34 \end{bmatrix}$

(d) $R = \begin{bmatrix} 1/2 & -3/10 \\ 0 & 1/5 \end{bmatrix}$ (e) $R = \begin{bmatrix} 5 & 0 \\ -3 & 2 \end{bmatrix}$

$$R = Q^T A = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 4 \\ 1 & -1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix}$$

4. Let A be an $m \times n$ matrix and let $\vec{b}$ be a vector in $Col(A)$. Which of the following statements may be false?

(a) There exists a vector $\vec{x}$ in $\mathbb{R}^n$ with $A\vec{x} = \vec{b}$.

(b) If A has linearly independent columns then $\det(A^T A) \neq 0$.

(c) $\vec{b}$ is perpendicular to every vector in the null space of A^T.

(d) If $A^T\vec{x} = A^T\vec{b}$ then $\vec{x}$ is contained in $Col(A)$.

(e) If $m > n$ then $\det(AA^T) = 0$.

(a) is true as b lies in $Col(A) = \text{range}(T_A)$

(b) is true since the normal equations have a unique solution

(c) true as $Col(A) = Nul(A^T)^\perp$

(d) May be false, eqn. says $x - b$ lies in $Nul(A^T) = Col(A)^\perp$

(e) true as $Nul(A^T) \neq \{0\}$.

5. Find the solution to the initial value problem

$$t\frac{dy}{dt} + y = t\sin t, \qquad y(\pi) = 1.$$

(a) $y = -\sin t - \pi\dfrac{\cos t}{t}$ (b) $y = -\cos t + \pi\dfrac{\sin t}{t}$

(c) $y = -\cos t + t\sin t$ (d) $y = -\cos t + \dfrac{\sin t}{t}$

(e) $y = -\cos t + 2\dfrac{\sin t}{t^2}$

ODE is linear, in standard form it's $y' + \left(\frac{1}{t}\right)y = \sin t$.

The integrating factor $\mu = e^{\int p\,dt} = e^{\ln t} = t$,

so a general solution is $y = \frac{1}{t}\left(\int t\sin t + C\right)$

$\Rightarrow y = -\cos t + \frac{\sin t}{t} + \frac{C}{t}$ $\quad y(\pi) = 1 \Rightarrow C = 0.$

6. Consider the initial value problem

$$\frac{dy}{dt} = y^2 - y - 2, \qquad y(0) = 0.$$

Which of the following describes the nature of the solution?

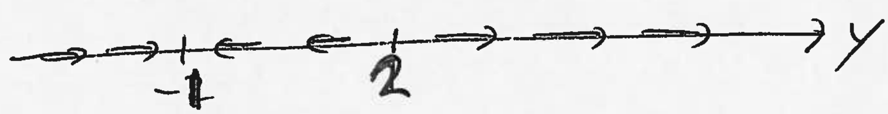
(a) $\lim_{t\to-\infty} y(t) = -1$; $\lim_{t\to\infty} y(t) = 2$; inflection point at $y = 1/2$

(b) $\lim_{t\to-\infty} y(t) = 2$; $\lim_{t\to\infty} y(t) = -1$; inflection point at $y = 1$

(c) $\lim_{t\to-\infty} y(t) = -2$; $\lim_{t\to\infty} y(t) = 1$; inflection point at $y = 0$

(d) $\lim_{t\to-\infty} y(t) = -1$; $\lim_{t\to\infty} y(t) = 2$; inflection point at $y = 1$

(e) $\lim_{t\to-\infty} y(t) = 2$; $\lim_{t\to\infty} y(t) = -1$; inflection point at $y = 1/2$

The phase line is 

So $\lim_{t\to\infty} y = -1$, $\lim_{t\to-\infty} y = 2$ (since start at $y(0)=0$).

Inflection pt. is when $(y^2-y-2)' = 0$ or $y = 1/2$

7. Let $y(t)$ be the unique solution of the initial value problem

$$\cos t \frac{dy}{dt} + t^2 y = \frac{1}{t-1} \qquad y(0) = C$$

What is the largest interval on which you can guarantee a solution y exists?

(a) $t > 0$ (b) $\frac{-\pi}{2} < t < 1$ (c) $t < 1$ (d) $-1 < t < 1$ (e) $\frac{-\pi}{2} < t < \frac{\pi}{2}$

In standard form, this is

$$y' + \underbrace{\left(\frac{t^2}{\cos t}\right)}_{p(t)} y = \underbrace{\frac{1}{\cos t\,(t-1)}}_{g(t)}.$$

$p(t)$ singular if $t = \pi/2 + n\pi$

$g(t)$ singular if $t = \pi/2 + n\pi$ or $t = 1$.

8. Find an implicit solution to the initial value problem

$$y + (x + e^y)\frac{dy}{dx} = 0, \qquad y(1) = 0.$$

(a) $x + e^y = 2$ (b) $y + xe^y = 1$ (c) $xy + e^y = 2$
(d) $xy + xe^y = 1$ (e) $xy + e^y = 1$

Set $M = y$, $N = x + e^y$

$\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = 1$ so ODE is exact.

$\frac{\partial \psi}{\partial x} = y \Rightarrow \psi = xy + h(y)$

$\frac{\partial \psi}{\partial y} = x + e^y \Rightarrow h'(y) = e^y \Rightarrow h = e^y + C$

So general soln. is $xy + e^y = C$, $y(1) = 0 \Rightarrow C = 1$

Part II: Partial credit questions (11 points each). Show your work.

9. Let $W = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where

$$\vec{v}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 6 \\ 0 \\ 3 \\ 1 \end{bmatrix}.$$

(a) Apply the Gram-Schmidt process to find an orthogonal basis $\mathcal{B}$ for W.

Set $y_1 = \begin{pmatrix} 3 \\ -3 \\ 0 \\ 1 \end{pmatrix}$, $y_2 = \left(\begin{pmatrix} 2 \\ 2 \\ -1 \\ 0 \end{pmatrix} - \frac{v_2 \cdot y_1}{y_1 \cdot y_1} \begin{pmatrix} 3 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right)$

$$= \begin{pmatrix} 2 \\ 2 \\ -1 \\ 0 \end{pmatrix} - 0 = \begin{pmatrix} 2 \\ 2 \\ -1 \\ 0 \end{pmatrix}$$

$$y_3 = \begin{pmatrix} 6 \\ 0 \\ 3 \\ 1 \end{pmatrix} - \frac{v_3 \cdot y_1}{y_1 \cdot y_1} \begin{pmatrix} 3 \\ -3 \\ 0 \\ 1 \end{pmatrix} - \frac{v_3 \cdot y_2}{y_2 \cdot y_2} \begin{pmatrix} 2 \\ 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 3 \\ 1 \end{pmatrix} - \frac{19}{19} \begin{pmatrix} 3 \\ -3 \\ 0 \\ 1 \end{pmatrix} - \frac{9}{9} \begin{pmatrix} 2 \\ 2 \\ -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 4 \\ 0 \end{pmatrix}$$

(b) Decide whether the vector $\vec{x} = \begin{bmatrix} 9 \\ 9 \\ 0 \\ 0 \end{bmatrix}$ is contained in W, and if it is then find the coordinate vector $[\vec{x}]_{\mathcal{B}}$ of $\vec{x}$ with respect to the basis $\mathcal{B}$ found in part (a).

$$\text{proj}_W \begin{pmatrix} 9 \\ 9 \\ 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 3 \\ -3 \\ 0 \\ 1 \end{pmatrix} + \frac{36}{9} \begin{pmatrix} 2 \\ 2 \\ -1 \\ 0 \end{pmatrix} + \frac{18}{18} \begin{pmatrix} 1 \\ 1 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \\ 0 \\ 0 \end{pmatrix}$$

so $\begin{pmatrix} 9 \\ 9 \\ 0 \\ 0 \end{pmatrix}$ lies in the subspace and its coordinates are

$$\begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}.$$

10. Let $A = \begin{bmatrix} 1 & -3 \\ 0 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 7 \\ 2 \\ 4 \end{bmatrix}$.

(a) Find the least square solution to the equation $A\vec{x} = \vec{b}$.

$$A^TA = \begin{pmatrix} 1 & 0 & 2 & 1 \\ -3 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 0 & 1 \\ 2 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ -3 & 10 \end{pmatrix}$$

$$A^Tb = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$$

So least square soln. satisfies $\begin{pmatrix} 6 & -3 \\ -3 & 10 \end{pmatrix} x = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$

or $x = \frac{1}{51} \begin{pmatrix} 10 & 3 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 9 \\ 4 \end{pmatrix} = \frac{1}{51} \begin{pmatrix} 102 \\ 51 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(b) Find the vector in the image of A which is closest to $\vec{b}$.

$$Ax = \begin{pmatrix} 1 & -3 \\ 0 & 1 \\ 2 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 4 \\ 2 \end{pmatrix}.$$

11. At time $t = 0$ a tank contains 50 gallons of water mixed with 10 kg. of salt. Over time well-mixed salt water drains from the tank at a rate of 2 gallons per day and is replenished with pure water which enters the tank at a rate of 3 gallons per day.

(a) Set up an initial value problem for the mass Q of salt remaining in the tank at time t.

$$\frac{dQ}{dt} = (\text{rate salt enters}) - (\text{rate salt exits})$$

$$= 0 - 2 \cdot \frac{Q}{50+t}, \quad \text{since pure water enters and the volume is } 50+t.$$

$$Q(0) = 10$$

(b) Solve the initial value problem to derive an explicit formula for Q.

The equation is separable, so

$$\int \frac{dQ}{Q} = -2 \int \frac{dt}{50+t}$$

so $\ln Q = -2\ln(50+t) + C$ or $Q = \dfrac{A}{(50+t)^2}$

$Q(0)=10 \Rightarrow 10 = \dfrac{A}{50^2} \Rightarrow A = 25000$

so $$Q = \frac{25000}{(50+t)^2}$$

12. **Find an implicit solution to the initial value problem**

$$y^2 + (xy+2)\frac{dy}{dx} = 0, \qquad y(0) = 1.$$

Set $M = y^2$, $N = xy + 2$

$M_y = 2y$, $N_x = y$, so eqn. not exact.

But $\dfrac{N_x - M_y}{M} = \dfrac{-1}{y}$, so there will be an integrating factor depending on y.

This satisfies $\mu' = \dfrac{-1}{y}\mu$

or $\ln \mu = -\ln y + C$. Set $\mu = \dfrac{1}{y}$.

Multiplying gives $y + \left(x + \dfrac{2}{y}\right)y' = 0$

which is exact.

The potential has $\Psi_x = y \Longrightarrow \Psi = xy + h(y)$

$\Psi_y = x + h'(y) \Longrightarrow$ set $h = 2\ln y$.

General soln. is $\Psi = xy + 2\ln y = C$. Initial condition gives $C = 0$.