Math 20580 Name:________________________

Final Exam Instructor:________________________

May 8, 2023 Section:________________________

Calculators are NOT allowed. You will be allowed 2 hours to do the test.

There are 20 multiple choice questions worth 7 points each. You will receive 10 points for following the instructions. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Sign the pledge. “On my honor, I have neither given nor received unauthorized aid on this Exam”:

1.	a	b	c	~~d~~	e	11.	a	b	~~c~~	d	e
2.	~~a~~	b	c	d	e	12.	a	~~b~~	c	d	e
3.	a	~~b~~	c	d	e	13.	~~a~~	b	c	d	e
4.	a	b	c	d	~~e~~	14.	a	~~b~~	c	d	e
5.	~~a~~	b	c	d	e	15.	a	b	c	d	~~e~~
6.	a	b	c	~~d~~	e	16.	a	~~b~~	c	d	e
7.	a	b	~~c~~	d	e	17.	a	b	c	d	~~e~~
8.	a	~~b~~	c	d	e	18.	~~a~~	b	c	d	e
9.	~~a~~	b	c	d	e	19.	~~a~~	b	c	d	e
10.	a	b	~~c~~	d	e	20.	a	~~b~~	c	d	e

1. Consider the bases $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$ for $\mathbb{R}^2$. Find the change of basis matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{\mathcal{P}}$ from $\mathcal{B}$ to $\mathcal{C}$.

(a) $\begin{bmatrix} 2 & 1 \\ 5 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -3 \\ 2 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 8 & -1 \\ -5 & 2 \end{bmatrix}$

(e) none of the above

$$\left[\begin{array}{cc:cc} -1 & -2 & 2 & -3 \\ 2 & 3 & 1 & 4 \end{array}\right] \xrightarrow{R_2 \to R_2 + 2R_1} \left[\begin{array}{cc:cc} -1 & -2 & 2 & -3 \\ 0 & -1 & 5 & -2 \end{array}\right]$$

$$\xrightarrow{R_2 \to -R_2} \left[\begin{array}{cc:cc} -1 & -2 & 2 & -3 \\ 0 & 1 & -5 & 2 \end{array}\right] \xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{cc:cc} -1 & 0 & -8 & 1 \\ 0 & 1 & -5 & 2 \end{array}\right]$$

$$\xrightarrow{R_1 \to -R_1} \left[\begin{array}{cc:cc} 1 & 0 & 8 & -1 \\ 0 & 1 & -5 & 2 \end{array}\right]$$

2. Which number is **not** an eigenvalue of the following matrix?

$$\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 4 & 0 & 0 \\ 0 & -1 & 1 & 3 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

(a) -3 (b) 4 (c) 1 (d) 2 (e) -5

$$\begin{vmatrix} 1-\lambda & 0 & 0 & 6 \\ 0 & 4-\lambda & 0 & 0 \\ 0 & -1 & 1-\lambda & 3 \\ 0 & 1 & 2 & -4-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 4-\lambda & 0 & 0 \\ -1 & 1-\lambda & 3 \\ 1 & 2 & -4-\lambda \end{vmatrix}$$

$$= (1-\lambda)(4-\lambda) \begin{vmatrix} 1-\lambda & 3 \\ 2 & -4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)(\lambda^2 + 3\lambda - 10)$$

$$= (1-\lambda)(4-\lambda)(\lambda+5)(\lambda-2)$$

3. If A is a 4×6 matrix of rank 2, what is the dimension of the orthogonal complement of the row space of A^T?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

$\text{Row}(A^T)^{\perp} = \text{nul}(A^T)$

A^T is 6×4 (4 columns)

Rank Nullity Thm. $\Rightarrow$ $\dim \text{nul}(A^T) = 4 - \text{rank}(A^T)$

$= 4 - \text{rank}(A)$

$= 2$

4. Consider the differential equation

$$(\overbrace{3x \sin y + 2e^y}^{M})dx + (\overbrace{x^2 \cos y + xe^y}^{N})dy = 0.$$

Find an integrating factor μ depending only on x that makes the equation exact.

(a) $\mu(x) = e^x$ (b) $\mu(x) = \sin x$ (c) $\mu(x) = x^2$ (d) $\mu(x) = \cos x$ (e) $\mu(x) = x$

$M_y = 3x\cos y + 2e^y$, $N_x = 2x\cos y + e^y$

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu = \frac{x\cos y + e^y}{x^2\cos y + xe^y}\mu = \frac{1}{x}\mu$$

So $\int \frac{d\mu}{\mu} = \int \frac{dx}{x}$ $\ln \mu = \ln x$

$\underline{\underline{\mu = x}}$

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x+2y \\ 3x+8y \end{bmatrix}. = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

Find a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that the matrix of T with respect to $\mathcal{B}$ (usually denoted by $[T]_{\mathcal{B}}$ or $\underset{\mathcal{B}\leftarrow\mathcal{B}}{T}$) is a diagonal matrix. $\Leftrightarrow$ $\boxed{\text{eigenbasis}}$

(a) $\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ (c) $\left\{\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix}\right\}$

(d) $\left\{\begin{bmatrix} 5 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ (e) $\left\{\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix}\right\}$

Std. matrix of T is $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$. Eigenvalues: $\begin{vmatrix} 3-\lambda & 2 \\ 3 & 8-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 11\lambda + 18 = 0 \Rightarrow \lambda = 2$ and $\lambda = 9$

$E_2 = \text{null}\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$ eigenvectors

$E_9 = \text{null}\begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$ form an eigenbasis

6. Which of the following sets of vectors is linearly independent?

(a) $\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}\right\}$ (c) $\left\{\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$

(d) $\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \\ 2 \\ 0 \end{bmatrix}\right\}$ (e) none of these

not scalar multiples of each other so independent

7. Let $\mathbb{P}_2$ denote the vector space of polynomials in x of degree at most 2, which has the basis $\mathcal{B} = \{1, x, x^2\}$. Consider the linear transformation

$$T : \mathbb{P}_2 \to \mathbb{P}_2, \qquad T(p(x)) = x^2 p''(x) + xp'(x) + p(x).$$

Compute the matrix of T with respect to $\mathcal{B}$ (denoted $[T]_{\mathcal{B}}$ or $\underset{\mathcal{B}\leftarrow\mathcal{B}}{T}$).

(a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 5 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & -1 \\ 5 & -1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 5 \\ 1 & 5 & 0 \end{bmatrix}$

$T(1) = 1$ $\qquad [T(1)]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$T(x) = x + x = 2x$ $\qquad [T(x)]_\beta = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$T(x^2) = x^2 \cdot 2 + x \cdot 2x + x^2 = 5x^2$ $\qquad [T(x^2)]_\beta = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$

8. Let $\mathcal{B} = \{1 - t, 2 + t^2, t - t^2\}$ be a basis for the vector space $\mathbb{P}_2$ of all polynomials in t of degree at most 2. Find the coordinate vector of $p = 5t - 2t^2$ with respect to $\mathcal{B}$.

(a) $[p]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 5 \\ -2 \end{bmatrix}$ (b) $[p]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$ (c) $[p]_{\mathcal{B}} = \begin{bmatrix} t-2 \\ 1 \\ 2 \end{bmatrix}$

(d) $[p]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ (e) $[p]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1/2 \\ 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ -1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -2 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 5 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & -2 \\ 0 & 2 & 1 & 5 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 3 & 9 \end{array}\right] \to \boxed{x_1 = -2}, \boxed{x_2 = 1}, \boxed{x_3 = 3}$$

9. Let

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

REF

If

r = dimension of the column space of A, $= 3$ (pivots)
s = dimension of the null space of A, $= 2$ (free variables)
t = dimension of the row space of A, then $= 3$

(a) $(r, s, t) = (3, 2, 3)$ (b) $(r, s, t) = (2, 3, 3)$ (c) $(r, s, t) = (2, 3, 2)$
(d) $(r, s, t) = (5, 0, 1)$ (e) $(r, s, t) = (3, 2, 4)$

10. The inverse of the matrix $\begin{bmatrix} -3 & -1 & 2 \\ 4 & 1 & -2 \\ 2 & 0 & -1 \end{bmatrix}$ is

(a) $\begin{bmatrix} -3 & 4 & 2 \\ -1 & 1 & 0 \\ 2 & -2 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 2 \\ -2 & -2 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 2 & 2 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 0 & -2 & -1 \end{bmatrix}$ (e) $\begin{bmatrix} -2 & -1 & 0 \\ 4 & 0 & 2 \\ 3 & 1 & -1 \end{bmatrix}$

$$\left[\begin{array}{ccc:ccc} -3 & -1 & 2 & 1 & 0 & 0 \\ 4 & 1 & -2 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 + 2R_3} \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 2 \\ 4 & 1 & -2 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 4R_1} \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 2 \\ 0 & 5 & -2 & -4 & 1 & -8 \\ 0 & 2 & -1 & -2 & 0 & -3 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_3}$$

$$\left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 2 & -1 & -2 & 0 & -3 \end{array}\right] \xrightarrow[R_1 \to R_1 + R_2]{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & -2 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to -R_3} \left[\begin{array}{c:ccc} I_3 & 1 & 1 & 0 \\ & 0 & 1 & -2 \\ & 2 & 2 & -1 \end{array}\right]$$

11. Find the solution of the initial value problem

$$\begin{cases} y'' - 5y' - 6y = 0, \\ y(0) = 0, \ y'(0) = 1. \end{cases}$$

(a) $\frac{1}{5}\left(e^{2t} - e^{3t}\right)$ (b) $e^{5t} - e^{6t}$ (c) $\frac{1}{7}\left(e^{6t} - e^{-t}\right)$
(d) $\frac{-1}{5}\left(e^{-2t} - e^{-3t}\right)$ (e) $\frac{-1}{7}\left(e^{t} - e^{-6t}\right)$

$m^2 - 5m - 6 = 0 \Rightarrow (m-6)(m+1) = 0 \Rightarrow FSS = \{e^{6t}, e^{-t}\}$

General soln. $y = c_1 e^{6t} + c_2 e^{-t}$

$y'(t) = 6c_1 e^{6t} - c_2 e^{-t}$

$$\begin{cases} c_1 + c_2 = 0 \\ 6c_1 - c_2 = 1 \end{cases}$$

$7c_1 = 1$

$\Rightarrow c_1 = 1/7$

$c_2 = -1/7$ $\Rightarrow y = \frac{1}{7}\left(e^{6t} - e^{-t}\right)$

12. Compute the Wronskian $W(y_1, y_2)$, where

$$y_1(t) = t\cos(t), \qquad y_2(t) = t\sin(t).$$

(a) 0 (b) t^2 (c) $t\sin(2t)$ (d) $\cos(2t)$ (e) $2t + 1$

$$\begin{vmatrix} t\cos t & t\sin t \\ \cos t - t\sin t & \sin t + t\cos t \end{vmatrix}$$

$= \cancel{t\cos t\sin t} + t^2\cos^2 t - \cancel{t\sin t\cos t} + t^2\sin^2 t$

$= t^2(\cos^2 t + \sin^2 t) = t^2$

constat . e^{2x}

13. Consider the differential equation $y'' - 3y' + 2y = e^{2x}$. By the method of undetermined coefficients, a particular solution will have the form

(a) Axe^{2x} (b) $(Ax^2 + B)e^{2x}$ (c) $2e^{2x}$ (d) $Ae^x + Be^{2x}$ (e) Ae^{2x}

FSS for $y'' - 3y' + 2y = 0$: $m^2 - 3m + 2 = 0$

$\Rightarrow m = 1$ or $m = 2$

$\Rightarrow FSS = \{e^x, e^{2x}\}$

$Y = x^s \cdot A \cdot e^{2x}$, $s = 0, 1, 2$ smallest so we don't get solution of homog. eqn.

$\Rightarrow s = 1$ and $Y = Axe^{2x}$

14. Find the solution of the initial value problem

$$\begin{cases} y' + 2y = 3e^x, \\ y(1) = 0 \end{cases} \leftarrow \text{linear}$$

(a) $(x - 1)e^x$ (b) $e^x - e^{3-2x}$ (c) x^2 (d) e^x (e) $(3e^x - 3)/2$

$\mu = e^{\int 2dx} = e^{2x}$

$$y = \frac{\int e^{2x} \cdot 3e^x \, dx}{e^{2x}} = \frac{e^{3x} + C}{e^{2x}}$$

$x = 1, y = 0 \Rightarrow \frac{e^3 + C}{e^2} = 0 \Rightarrow \boxed{C = -e^3}$

$\Rightarrow y = e^x - \frac{e^3}{e^{2x}} = \boxed{e^x - e^{3-2x}}$

15. If x is a real number, what are the possible values for the rank of the matrix

$$\begin{bmatrix} 1 & 1 & x \\ 1 & x & 0 \\ x & 0 & 0 \end{bmatrix}?$$

(a) 1, 2 (b) 0 (c) 0, 1, 2, 3 (d) 0, 1, 2 (e) 2, 3

$$\det = x \cdot \begin{vmatrix} 1 & x \\ x & 0 \end{vmatrix} = x \cdot (-x^2) = -x^3$$

So if $\boxed{x \neq 0}$ then $\det \neq 0 \Rightarrow \boxed{\text{rank} = 3}$

If $x = 0$, matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has $\boxed{\text{rank } 2}$

orthogonal

16. Consider the subspace W of $\mathbb{R}^3$ with basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ and the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

$\vec{x}_1$ " $\vec{x}_2$ "

Which of the following is the orthogonal projection of $\mathbf{v}$ onto the subspace W?

(a) $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$ (c) $\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ (e) $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

$$\text{proj}_W(\vec{v}) = \frac{\vec{v} \cdot \vec{x}_1}{\vec{x}_1 \cdot \vec{x}_1} \vec{x}_1 + \frac{\vec{v} \cdot \vec{x}_2}{\vec{x}_2 \cdot \vec{x}_2} \vec{x}_2$$

$$= \frac{-2}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{9}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

17. Find a least-squares solution of the equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

(a) $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{-15}{21} \\ \frac{5}{21} \end{bmatrix}$ (c) $\begin{bmatrix} \frac{2}{5} \\ \frac{1}{2} \end{bmatrix}$ (d) $\begin{bmatrix} \frac{21}{25} \\ \frac{-10}{25} \end{bmatrix}$ (e) $\begin{bmatrix} \frac{25}{21} \\ \frac{-10}{21} \end{bmatrix}$

$A^TA = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$

$A^T\vec{b} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

So $\hat{x} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \frac{1}{21}\begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \frac{1}{21}\begin{bmatrix} 25 \\ -10 \end{bmatrix}$

18. The function $y_1(x) = x^2$ is a solution of the differential equation

$$x^2y'' + 2xy' - 6y = 0, \ x > 0.$$

Using the method of reduction of order, find a second solution $y_2(x)$ for this equation.

(a) $1/x^3$ (b) e^{3x} (c) $x\ln(x)$ (d) $x^2 + 2x - 6$ (e) x^2e^x

$-6/ \quad y_2 = v \cdot x^2$

$2x/ \quad y_2' = v \cdot 2x + v' \cdot x^2$

$x^2/ \quad y_2'' = v \cdot 2 + 2v' \cdot 2x + v'' \cdot x^2$

$0 = v' \cdot (4x^3 + 2x^3) + v'' \cdot x^4$

divide by x^3 $\Rightarrow$ $6v' + x \cdot v'' = 0$

$\Rightarrow w = v'$

$\frac{dw}{dx} = -\frac{6}{x} w$

$\Rightarrow w = e^{\int -6/x \, dx} = e^{-6\ln x} = x^{-6}$

$\Rightarrow v = \int x^{-6} dx = \frac{x^{-5}}{-5}$

$\Rightarrow y_2 = v \cdot x^2 = \frac{x^{-3}}{-5}$ (or constant multiple)

19. Given that $y_1 = t$ and $y_2 = t^{-1}$ form a fundamental set of solutions for the differential equation $y'' + t^{-1}y' - t^{-2}y = 0$, the variation of parameters method gives a particular solution Y to the corresponding nonhomogeneous equation

$$y'' + t^{-1}y' - t^{-2}y = \underbrace{t^{-1}}_{g(t)}, \quad t > 0,$$

$$W(y_1, y_2) = \begin{vmatrix} t & t^{-1} \\ 1 & -t^{-2} \end{vmatrix} = -2t^{-1}$$

of the form

$$Y = tu_1(t) + t^{-1}u_2(t).$$

Which of the following functions can be $u_1(t)$?

(a) $u_1(t) = \dfrac{1}{2}\ln t$ (b) $u_1(t) = t\ln t$ (c) $u_1(t) = -\dfrac{1}{2}t^2$
(d) $u_1(t) = t^2\ln t$ (e) $u_1(t) = 2t + 5t^{-1}$

$$u_1 = \int \frac{-t^{-1} \cdot t^{-1}}{-2t^{-1}}\,dt = \frac{1}{2}\int \frac{1}{t}\,dt = \frac{1}{2}\ln t$$

20. Which formula describes implicitly the solution of the initial value problem

$$\frac{dy}{dx} = \frac{x+1}{x \cdot (y^2+1)}, \quad y(1) = 0, \qquad x > 0.$$

(a) $3x + \ln x - 3y = 3$ (b) $y^3 + 3y = 3x + 3\ln x - 3$ (c) $x - \ln x - y^3 = 1$
(d) $y^3 + y = 3x - 3$ (e) $-y^3 + 3y = 3x + 3\ln x$

$$\int (y^2+1)\,dy = \int \frac{x+1}{x}\,dx = \int \left(1 + \frac{1}{x}\right)dx$$

$$\frac{y^3}{3} + y = x + \ln x + C$$

$x=1, y=0$ $\quad 0 = 1 + 0 + C \Rightarrow \boxed{C=-1}$

$$y^3 + 3y = 3x + 3\ln x - 3$$