Math 20610: Homework 10, Additional Exercises
Due Wednesday, Nov. 14.

Let $V$ and $W$ be finite dimensional vector spaces over $\mathbb{R}$. Let $T : V \rightarrow W$ be a linear transformation. Fix a basis $B_V = \{v_1, v_2, \ldots, v_n\}$ of $V$, and a basis $B_W = \{w_1, w_2, \ldots, w_m\}$ of $W$. Let $M = [T]_{B_V, B_W} = \mathcal{M}_{B_V, B_W}(T) \in \text{Mat}_{m,n}$ be the matrix corresponding to $T$ with respect to the bases $B_V$ and $B_W$.

Then we have

$$T : V \rightarrow W$$

$$\mathcal{M}_{B_V} = \begin{bmatrix} \cdot \end{bmatrix}_{B_V}$$

$\downarrow$

$$\mathcal{M}_{B_W} = \begin{bmatrix} \cdot \end{bmatrix}_{B_W}$$

$$\mathcal{M}_{B_V, B_W}(T) = M : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

And from the matrix equation, we have that the matrix corresponding to $T$ with respect to the $B_V$ and $B_W$ bases is given by

$$M = [T]_{B_V, B_W} = \begin{bmatrix} [T(v_1)]_{B_W} & [T(v_2)]_{B_W} & \cdots & [T(v_n)]_{B_W} \end{bmatrix}.$$

1. Let $T$ be the linear transformation defined as follows:

$$T : \mathbb{P}_2 \rightarrow \text{Mat}_{2 \times 2}$$

$$a + bt + ct^2 \mapsto \begin{pmatrix} a + c & -2a + b - 3c \\ a - 2b + 3c & b + c \end{pmatrix}$$

Let $B = (1, t, t^2)$ be a basis for $\mathbb{P}_2$ and let $C = (E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2})$ be a basis for $\text{Mat}_{2 \times 2}$ where $E_{i,j}$ is the matrix with a 1 in the $i$-th row of the $j$-th column and zeros elsewhere.

(a) Give the matrix $M$ corresponding to $T$ with respect to the bases $B$ and $C$.

(b) What is $\text{Null } M$? What is $\text{Null } T$? That is, give a basis for each of these subspaces.

(c) What is $\text{Im } M$? What is $\text{Im } T$? That is, give a basis for each of these subspaces.

(d) What is the rank of $T$?

2. Let $T$ be the linear transformation defined as follows:

$$T : \mathbb{P}_3 \rightarrow \mathbb{P}_3$$

$$p(t) \mapsto 2p''(t) + p'(t) + tp(0).$$
Let $\mathcal{B} = (1, t, t^2, t^3)$ be a basis for $\mathbb{P}_3$.

(a) Give the $\mathcal{B}$-matrix $M$ for $T$.

(b) What is Null $M$? What is Null $T$? That is, give a basis for each of these subspaces.

(c) What is $\text{Im } M$? What is $\text{Im } T$? That is, give a basis for each of these subspaces.

(d) What is the rank of $T$?

3. Let $T$ be the linear transformation defined as follows:

$$T : \mathbb{P}_3 \rightarrow \mathbb{P}_4$$

$$p(t) \mapsto t^2p''(t) + tp'(t) + tp(t).$$

Let $\mathcal{B} = \{1, t, t^2, t^3\}$ be a basis for $\mathbb{P}_3$, and $\mathcal{C} = \{1, t, t^2, t^3, t^4\}$ a basis for $\mathbb{P}_4$.

(a) Give the matrix representation $M$ of $T$ with respect to the basis $\mathcal{B}$ and $\mathcal{C}$.

(b) What is Null $M$? What is Null $T$? That is, give a basis for each of these subspaces.

(c) What is $\text{Im } M$? What is $\text{Im } T$? That is, give a basis for each of these subspaces.

(d) What is the rank of $T$?