Math 70220: Homework 3 – Due Fri., Feb. 21, 2025.

Do the following problems from Humphreys, Chapter II, Section 4:

Section 4, p. 20.: #1, 2, 3.

In addition, do the following exercises:

1. In certain models in physics, namely String Theory, or more generally, Conformal Field Theory, particles have corresponding "fields" describing how they interact with other particles. (Here the word "field" is used in a completely different way than in the algebraic sense we have been using it.) If we restrict to just a two-dimensional complex analytic theory, then the particle fields are similar to functions in a complex variable $z \in \mathbb{C}$. So for the purposes of this problem, denote a "particle field" by a function in one variable that is infinitely differentiable. Denote the space of such functions as $C^\infty(U)$, for U some open set in $\mathbb{C}$.

Recall from Homework 1, the Witt Lie algebra $W = \text{span}_\mathbb{C}\{-z^{n+1}\frac{d}{dz} \mid n \in \mathbb{Z}\}$ and note that this Lie algebra lies in $\mathfrak{gl}(C^\infty(U))$, for U an open set in $\mathbb{C} \smallsetminus \{0\}$.

The operator $L_0 = -z\frac{d}{dz}$ models the "energy" for the theory in a certain sense. In many physical theories, we are used to the vector space of particle fields decomposing into eigenspaces for the operator L_0.

In most models (such as quantum mechanics) the eigenvalues for L_0 are discrete and integer or half-integer. But in some disordered systems and in some new models of current interest, L_0 has generalized eigenvectors not just eigenvectors. These theories are called "logarithmic theories". Exercise #1 will let you see why they were given this name.

(a) Set up a differential equation and solve it to determine the eigenvalues and eigenvectors for L_0 in $C^\infty(U)$.

(b) Now assume that there is a subspace $V \subset \mathbb{C}^\infty(U)$ of the space of particle fields for which L_0 restricted to V is given by

$$[L_0]_\mathcal{B} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

in the basis for V given by a particular $\mathcal{B} = (f(z),\ g(z))$. That is $f(z) \in \text{Null}(L_0 - \lambda I)$ is an eigenvector with eigenvalue $\lambda \in \mathbb{C}$. So fix such an $f(z)$. And $g(z)$ is not an eigenvector, but a generalized eigenvector in $\text{Null}(L_0 - \lambda I)^2$.

What is $L_0 g(z)$ with respect to this basis $\mathcal{B} = (f(z),\ g(z))$?

(c) Write a linear differential equation involving $g(z)$ and $f(z)$ from part (b).

(d) Solve the linear differential equation from part (c) to find the general for of the solution for $g(z)$.

2. In Homework #2, you showed that the transpose map was not a Lie algebra homomorphism, but instead a Lie algebra anti-homomorphism. Let $V = \mathfrak{gl}(n, \mathbb{F})$, for $n > 1$ and char $\mathbb{F} \neq 2$ and again consider the transpose map

$$\begin{aligned} T : V &\longrightarrow V \\ M &\mapsto M^t \end{aligned}$$

(a) What is T^2?

(b) What is the minimal polynomial of T?

(c) What are the eigenvalues for T?

(d) What is the space of eigenvectors for each of the eigenvalues of T?

(e) Is T diagonalizable? Prove your claim.

3. Recall the vector cross product in $\mathbb{R}^3$

$$\mathbf{x} \times \mathbf{y} = (x_2y_3 - x_3y_2,\ x_3y_1 - x_1y_3,\ x_1y_2 - x_2y_1),$$

for $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$.

(a) Verify that this binary structure is bilinear.

(b) Verify that

$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = (\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z}$$

for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$ and where $\cdot$ denotes the usual dot product in $\mathbb{R}^3$.

(c) Verify that $\mathbb{R}^3$ under the cross product is a Lie algebra.

(d) Give the structure constants for the standard basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ for this Lie algebra.

4. Let L be the 3-dimensional Lie algebra over $\mathbb{C}$ with basis (a, b, c) and Lie bracket defined by

$$[a, b] = c, \qquad [b, c] = a, \qquad \text{and} \qquad [c, a] = b$$

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(a) Note that L is the "complexification" of the 3-dimensional Lie algebra over $\mathbb{R}$ given in Exercise #3. Show that L is isomorphic as a Lie algebra over $\mathbb{C}$ to the subalgebra of $\mathfrak{gl}(3, \mathbb{C})$ consisting of all 3×3 antisymmetric matrices with entries in $\mathbb{C}$. (That is matrices that satisfy $M^T = -M$.)

(b) Find an explicit isomorphism to show that in fact $L \cong \mathfrak{sl}(2, \mathbb{C})$.

5. Let $\mathbb{F}$ be a field of characteristic other than 2. Let $A = \mathrm{span}_{\mathbb{F}}\{x, y, z\}$ be the $\mathbb{F}$-algebra with $\mathbb{F}$-algebra structure given by the binary operation $[\cdot,\cdot] : A \times A \longrightarrow A$ determined by the brackets

$$[x, y] = x, \qquad [x, z] = y, \qquad [y, z] = 0,$$

extended linearly, along with $[a, a] = 0$ for all $a \in A$.

(a) Consider the linear map

$$\begin{aligned} ad : A &\longrightarrow \mathfrak{gl}(A) \\ a &\mapsto ad\, a. \end{aligned}$$

Give matrix representations for $ad\, x$, $ad\, y$, and $ad\, z$ with respect to the basis (x, y, z).

(b) Is ad an injective map?

(c) Is ad an $\mathbb{F}$-algebra homomorphism with respect to the standard Lie algebra structure on $\mathfrak{gl}(A)$?

(d) Is A a Lie algebra?