Math 70220: Homework 5 – Due Friday, Mar. 6, 2020.

Do the following problems from Humphreys, Chapter II:

Section 5, p. 24: #1 – 8.

Also do the following problem:

1. Let $L = \mathfrak{sp}(2n, \mathbb{F}) = C_n$ be the classical Lie algebra as defined on p. 3 of Humphreys. Thus $X \in L$ has a block decomposition

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A, B, C, D$ are $n \times n$ submatrices. Then as in Humphreys, $L = C_n$ is defined to be $L = \{X \in \mathfrak{gl}(2n, \mathbb{F}) \mid X^t J + J X = 0\}$ for

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

which, by your previous homework, is a subalgebra of $\mathfrak{gl}(2n, \mathbb{F})$.

(a) Prove that $X \in L$ if and only if $D = -A^t$, $B^t = B$ and $C^t = C$.

(b) Consider the following subspaces of $L$:

$h$ = $\{X \mid A$ diagonal, $B = C = 0\}$

$n^+ =$ $\{X \mid A$ strictly upper triangular, $B^t = B$, $C = 0\}$

$n^- =$ $\{X^t \mid X \in n^+\}.$

Let $h^*$ denote the dual space of $h$ (as a vector space), and let $\epsilon_i \in h^*$ be defined by

$$\epsilon_i \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} = a_i$$

if $A$ is the diagonal matrix with entries $a_1, \ldots, a_n$.

Set

$$\Phi_+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\} \cup \{\epsilon_i + \epsilon_j \mid 1 \leq i \leq j \leq n\}.$$ 

Find a basis $\{E_{\alpha} \mid \alpha \in \Phi_+\}$ for $n^+$ such that

$$[H, E_{\alpha}] = \alpha(H) E_{\alpha} \quad \text{for all } H \in h.$$
(c) What is $[H, E^t_\alpha]$ for each basis element $E_\alpha$ of $n^+$ you found in part (b)?

(d) What is $[E_\alpha, E^t_\alpha]$ for $\alpha \in \Phi_+$?

(e) For fixed $\alpha \in \Phi_+$, what is the dimension of the Lie subalgebra of $L$ generated by $E_\alpha$ and $E^t_\alpha$, and can you name a more familiar Lie algebra that this subalgebra is isomorphic to?