Math 30820: Homework 7 – Due 03/27/2024.

1. Let R be a ring, M and N left R-modules, and $\varphi : M \longrightarrow N$ an R-module homomorphism.

(a) Prove that $\ker \varphi$ is a submodule of M and $\operatorname{Im} \varphi$ is a submodule of N.

(b) Prove the First Isomorphism Theorem for R-modules, i.e. that $M/\ker\varphi$ is isomorphic to $\operatorname{Im} \varphi$ as R-modules.

2. As a warm-up to Exercise #2.1 in Artin: Consider the ideal $I = (2, x) \subset \mathbb{Z}[x]$ as a $\mathbb{Z}[x]$-module.

(a) Show that although $(2, x)$ is a spanning list for I, it is not a linearly independent spanning list, i.e. not a basis for I.

(b) Show that the ideal I is not free as a $\mathbb{Z}[x]$-module.

3. Let R be a ring.

(a) Let $M_1, M_2, \ldots, M_k$ be R-modules, and let $N_1, N_2, \ldots, N_k$ be submodules of $M_1, M_2, \ldots, M_k$, respectively, i.e. $N_i \subseteq M_i$ for $i = 1, \ldots, k$. Prove that as R-modules

$$(M_1 \oplus M_2 \oplus \cdots \oplus M_k)/(N_1 \oplus N_2 \oplus \cdots \oplus N_k) \cong M_1/N_1 \oplus M_2/N_2 \oplus \cdots \oplus M_k/N_k.$$

(b) Show that the result above fails if N_i is not a submodule of M_i for each $i = 1, \ldots, k$.

4. Express $\mathbb{Z}_4 \times \mathbb{Z}_8 \times \mathbb{Z}_3$ three different ways as a $\mathbb{Z}$-module isomorphic to a quotient of a free $\mathbb{Z}$-module by a submodule.