Do the following problems from Humphreys, Chapter II:

Section 7, p. 34: #1, 2, 5, 6.

Section 8, p. 40: #1, 2, 3. (Note: Exercises for $A_n, C_n, D_n$ in HW 4-6 and Exercise #2 for $B_n$ below give you the answers to Exercise #8.2 on p. 40 of Humphreys.)

Notes on Exercises from the book that were not assigned:
· We basically did Exercise #7.3 on p. 34 of Humphreys in “class”, i.e., in the video for Section 7.
· Exercise #1 below, is essentially Exercise #7.4 on p. 34 of Humphreys.
· We did an example in the video for Section 7 that is equivalent to the one outlined in Exercise #7.7 on p. 34 of Humphreys.

Also do the following two problems:

1. Let $\mathbb{F}$ be a field of characteristic zero and let $\mathbb{F}[X,Y]$ be the vector space of polynomials in the two commuting variables $X$ and $Y$.

Define the representation $\varphi$ of $\mathfrak{sl}(2,\mathbb{F})$ on $\mathbb{F}[X,Y]$ by letting

$$
\varphi(x) = X \frac{\partial}{\partial Y}, \quad \varphi(y) = Y \frac{\partial}{\partial X}, \quad \text{and} \quad \varphi(h) = X \frac{\partial}{\partial X} - Y \frac{\partial}{\partial Y},
$$

and extending $\varphi$ linearly to all of $\mathfrak{sl}(2,\mathbb{F})$.

(a) Verify that $\varphi$ as defined above is indeed a representation of $\mathfrak{sl}(2,\mathbb{F})$, i.e., that $\mathbb{F}[X,Y]$ is an $\mathfrak{sl}(2,\mathbb{F})$-module.

(b) Determine the eigenvalues and eigenvectors of $\varphi(h)$.

(c) Prove that even though $\mathbb{F}[X,Y]$ is infinite dimensional, it is completely reducible as an $\mathfrak{sl}(2,\mathbb{F})$-module by decomposing $\mathbb{F}[X,Y]$ into a sum of irreducible $\mathfrak{sl}(2,\mathbb{F})$-submodules.

2. Let $L = \mathfrak{so}(2n+1,\mathbb{F}) = B_n$ be the classical Lie algebra as defined on p. 3 of Humphreys. Thus $X \in L$ has a block decomposition

$$
X = \begin{pmatrix}
a & x & y \\
u & A & B \\
v & C & D
\end{pmatrix}
$$

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where $A, B, C, D$ are $n \times n$ submatrices, $a \in \mathbb{F}$, $x, y$ are $1 \times n$ matrices, and $u, v$ are $n \times 1$ matrices. Then as in Humphreys, $L = B_n$ is defined to be

$$L = \{ X \in \mathfrak{gl}(2n+1, \mathbb{F}) \mid X^t J + JX = 0 \}$$

for

$$J = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & I_n \\
0 & I_n & 0
\end{pmatrix}.$$

(a) Prove that $X \in L$ if and only if $a = 0$, $D = -A^t$, $B^t = -B$ and $C^t = -C$, $u = -y^t$ and $v = -x^t$.

(b) Consider the following subspaces of $L$:

$$h = \{ X \mid A \text{ diagonal, } B = C = 0, x = y = 0 \}$$

$$n^+ = \{ X \mid A \text{ strictly upper triangular, } B^t = -B, C = 0, x = 0 \}$$

$$n^- = \{ X^t \mid X \in n^+ \}.$$

Let $\mathfrak{h}^*$ denote the dual space of $\mathfrak{h}$ (as a vector space), and let $\epsilon_i \in \mathfrak{h}^*$ be defined by

$$\epsilon_i \begin{pmatrix} 0 & 0 & 0 \\
0 & A & 0 \\
0 & 0 & -A \end{pmatrix} = a_i$$

if $A$ is the diagonal matrix with entries $a_1, \ldots, a_n$.

Set

$$\Phi_+ = \{ \epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n \} \cup \{ \epsilon_i + \epsilon_j \mid 1 \leq i < j \leq n \} \cup \{ \epsilon_i \mid 1 \leq i \leq n \}.$$

Find a basis $\{ E_\alpha \mid \alpha \in \Phi_+ \}$ for $n^+$ such that

$$[H, E_\alpha] = \alpha(H)E_\alpha \quad \text{for all } H \in \mathfrak{h}.$$

(c) What is $[H, E_\alpha]$ for each basis element $E_\alpha$ of $n^+$ you found in part (b)?

(d) What is $[E_\alpha, E_\beta]$ for $\alpha \in \Phi_+$?

(e) For fixed $\alpha \in \Phi_+$, what is the dimension of the Lie subalgebra of $L$ generated by $E_\alpha$ and $E_\alpha^t$, and can you name a more familiar Lie algebra that this subalgebra is isomorphic to?