Math 70220: Homework 8 – — Due Wednesday, Apr. 30, 2025.

Do the following problems from Humphreys, Chapter III:

Section 9, p. 46: #3, 4, 5, 6.
Section 10, p. 54: #9.
Section 11, p. 63: #3.

Also the following problems discuss semi-direct product constructions and a proof that there are an infinite number of isomorphism classes of Lie algebras of dimension 3.

1. Let V be a vector space over a field $\mathbb{F}$, and let $\varphi : V \longrightarrow V$ be a linear transformation. Let L be a 1-dimensional extension of V, denoted $L = V \oplus \mathbb{F}x$. Define an $\mathbb{F}$-algebra structure on L via

$$[y, z] = 0 \quad \text{and} \quad [x, y] = \varphi(y),$$

for all $y, z \in V$, extended linearly, and assuming $[\cdot, \cdot]$ is skew-symmetric. We call this $\mathbb{F}$-algebra L_φ.

(a) Show L_φ is a Lie algebra.

(b) Show $\dim L^1_\varphi = \operatorname{rank} \varphi$, where $L^1_\varphi = [L_\varphi, L_\varphi]$.

2. In the setting of Exercise 1, let V be 2-dimensional with $V = \operatorname{span}_{\mathbb{F}}\{y, z\}$, so that $L_\varphi = \operatorname{span}_{\mathbb{F}}\{x, y, z\}$ is 3-dimensional with bracket given as in Exercise 1, where $\varphi(y) = y$ and $\varphi(z) = \mu z$ for some $\mu \in \mathbb{F} \setminus \{0\}$, and denote this L_φ by L_μ.

(a) Prove that $L_\mu \cong L_\nu$ if and only if either $\mu = \nu$ or $\mu = \nu^{-1}$.

(b) What can you conclude about the number of isomorphism classes of Lie algebras of dimension 3?

3. The constructions in Exercise 1 and 2 are examples of **semi-direct products**. We now give the general construction.

Suppose that I is an ideal of a Lie algebra L and there is a subalgebra S of L that is a direct sum complement to I as a vector space.

(a) Show that the map

$$\begin{aligned} \theta : S &\longrightarrow \mathfrak{gl}(I) \\ s &\mapsto \theta(s) : x \mapsto [s,x] \qquad \text{for } x \in I \end{aligned}$$

is a Lie algebra homomorphism from S into Der I.

We call this construction the **semi-direct product of I by S** which is denoted

$$L = S \ltimes_\theta I.$$

(b) Conversely, show that given Lie algebras S and I and a Lie algebra homomorphism $\theta : S \longrightarrow \text{Der } I$, then the vector space $S \oplus I$ may be made into a Lie algebra by defining

$$[(s,x),(r,y)] = ([s,r],[x,y] + \theta(s)x - \theta(r)y)$$

for $s, r \in S$ and $x, y \in I$, and that this resulting Lie algebra is a semi-direct product of I by S as given in part (a).

(c) Show that the Lie algebras in Exercise 1 may be constructed as semi-direct products.