Hopf actions

& algebras in tensor categories

QuaSy-Con II

University of Notre Dame

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Introductory Talk:

≡ Questions welcome throughout ≡

I. Hopf Actions & algebras in tensor categories

II. Quivers & structure of fin. dim. $\mathbb{k}$-algebras

III. Tensor algebras in (finite) tensor categories

I. Hopf Actions & Algebras in Tensor Categories

$\mathbb{k}$ is a field & everything $\mathbb{k}$-linear

Def: A Hopf algebra is an associative $\mathbb{k}$-algebra H with additional structure:

$\Delta : H \longrightarrow H \otimes H$ (comultiplication)

$\varepsilon : H \longrightarrow \mathbb{k}$ (counit)

satisfying some compatibility axioms.

Intuition: For a noncommutative ring R and $M, N \in R\text{-Mod}$, there is no canonical way to form "$M \otimes N$" in R-Mod, and no (nonzero) "trivial" R-Mod.

The extra structure of a Hopf algebra fixes this:

- $M, N \in H\text{-Mod} \xRightarrow{\text{use } \Delta} M \otimes_{\mathbb{k}} N \in H\text{-Mod}$
- $\xRightarrow{\text{use } \varepsilon} \mathbb{k} \in H\text{-Mod}$

Ex: G : group, $H = \Bbbk G$ group algebra.

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1 \quad \forall g \in G.$$

- Then $M, N \in \Bbbk G\text{-Mod} \Rightarrow M \otimes_{\Bbbk} N \in \Bbbk G\text{-Mod}$ via $g \cdot (m \otimes n) = (g \cdot m) \otimes (g \cdot n)$
- $\Bbbk G$ acts on $\Bbbk$ "trivially" via
$$g \cdot \alpha = \varepsilon(g)\alpha = \alpha \quad \forall g \in G \;\; \forall \alpha \in \Bbbk$$

Ex: $\mathfrak{g}$: Lie algebra, $H = U(\mathfrak{g})$ envel. algebra

$$\Delta(x) = 1 \otimes x + x \otimes 1, \quad \varepsilon(x) = 0 \quad \forall x \in \mathfrak{g}$$

- Then $M, N \in U(\mathfrak{g})\text{-Mod} \Rightarrow M \otimes_{\Bbbk} N \in U(\mathfrak{g})\text{-Mod}$ via $x \cdot (m \otimes n) = m \otimes (x \cdot n) + (x \cdot m) \otimes n$
- $\mathfrak{g}$ (or $U(\mathfrak{g})$) acts on $\Bbbk$ "trivially" via
$$x \cdot \alpha = \varepsilon(x)\alpha = 0 \quad \forall x \in \mathfrak{g}, \forall \alpha \in \Bbbk$$

- "Symmetries" of an algebra can be formalized by having a group act on that algebra.

<u>Ex:</u> Symmetric group S_n acts on poly ring $\mathbb{k}[x_1, x_2, \dots, x_n]$ by permuting variable labels.

- This generalizes nicely to Hopf algebras:

<u>Def</u>: Let A be an assoc. $\mathbb{k}$-alg.
An <u>action of a Hopf algebra H on A</u> is:
an H-Module structure on A
(the action, how $h \cdot a \in A$ $\forall h \in H, a \in A$)
which:
- "respects" multiplication:
  $A \otimes A \to A$ is H-Mod. homom.
- acts "trivially" on scalars:
  $\mathbb{k} \to A$, $1_{\mathbb{k}} \mapsto 1_A$ is H-Mod hom

Key idea & examples:

"Extra structure" on a $\Bbbk$-algebra A can often be formalized as a Hopf algebra action on A.

Ex: Action of group G on A (by automorphisms)

$\Updownarrow$ equiv. data

Hopf Action of $\Bbbk G$ on A

Ex: Action of Lie algebra $\mathfrak{g}$ on A (by derivations)

$\Updownarrow$ equiv data

Hopf action of $U(\mathfrak{g})$ on A

Ex: Grading of A by finite group G

$\Updownarrow$ equiv. data

Hopf action of $(\Bbbk G)^*$ on A

One can take fancier Hopf algebras like quantum groups $U_q(\mathfrak{g})$ as well...

"<u>Quantum symmetries</u>" of an algebra in this talk will mean an action of a Hopf algebra.

Algebras in tensor categories

"Def:" A tensor category $\mathcal{C}$ in this talk can be thought of as an abelian category with an additional operation

$$\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C},$$

admitting some nice features like

- unit object $\mathbb{1} \in \mathcal{C}$ s.t. $\mathbb{1} \otimes X \cong X \cong X \otimes \mathbb{1}$ $\forall X \in \mathcal{C}$ & $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \Bbbk$

- associativity isomorphisms $a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ $\forall X, Y, Z \in \mathcal{C}$

- dual objects $X^* \in \mathcal{C}$ $\forall X \in \mathcal{C}$

We also assume tensor categories in this talk are finite, meaning equiv. to fin. dim. reps of a fin. dim alg. (unrelated to $\otimes$ operation).

Examples

- $\mathcal{C} = \mathrm{vec}$, fin. dim. $\mathbb{k}$-vec spaces.

- $\mathcal{C} = \mathrm{rep}(H)$ for a Hopf algebra H.

- $\mathcal{C} = \mathrm{vec}_G^\omega$ where :
  - G finite group
  - $\omega : G \times G \times G \longrightarrow \mathbb{k}$ 3-cocycle
  - objects & morphisms are G-graded fin. dim. vec. spaces
  - associativity twisted by ω : for X, Y, Z in degrees $g, h, m \in G$, resp., $(X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ is mult. by scalar $\omega(g, h, m) \in \mathbb{k}$.

- Fibonacci Category $\mathcal{C}$, semisimple with two simple objects $\mathbb{1}, \tau \in \mathcal{C}$ s.t. $\tau \otimes \tau \cong \mathbb{1} \oplus \tau$.

Note that while objects in first 3 examples are vectors spaces by def, this is not inherently part of structure of general tensor cat. $\mathcal{C}$.

In fact, if we try to think of objects in Fibonacci category as vector spaces, we find:

$$(\dim \tau)^2 = \dim \tau + 1$$

Therefore, all concepts in tensor categories must be defined in terms of morphisms between objects, NOT internal structure of objects e.g. "elements".

<u>Def</u>: An <u>algebra</u> in $\mathcal{C}$ is object $A \in \mathcal{C}$ along with:

- "multiplication" $A \otimes A \to A$ in $\mathcal{C}$
- "scalars" $\mathbb{1} \to A$ in $\mathcal{C}$

satisfying natural axioms.

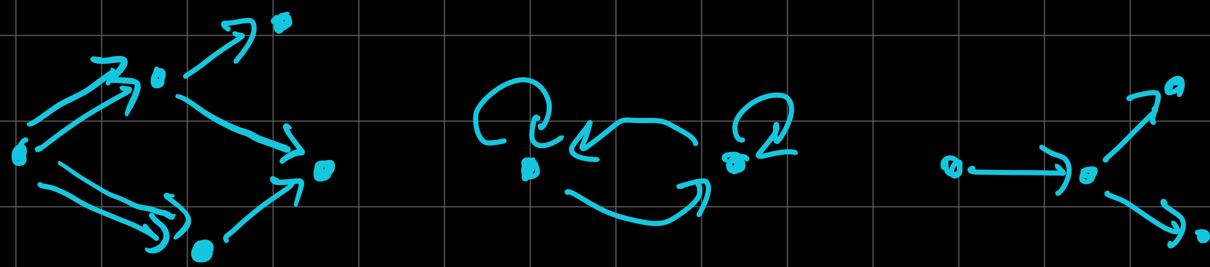
<u>Key Example</u>: For Hopf algebra H, an algebra A in $\mathcal{C} = \mathrm{rep}(H)$ is defined by the same data as a Hopf action of H on A.

<u>Why?</u> The more abstract framework sometimes makes it easier to see connections to related concepts, e.g. <u>coactions</u> of H on A, variation of $\mathcal{C}$, etc.

- Usually done through <u>module categories</u> $\mathcal{M}$ over $\mathcal{C}$, outside scope of this talk.

II. Quivers & structure of fin. dim. $\Bbbk$-algebras

Def: A *quiver* Q is a (finite) directed graph, allowing loops, parallel edges, etc.



They are interesting in algebra/rep theory because:

Def: The *path algebra* $\Bbbk Q$ of a quiver Q is:

- vector space with basis $\left\{ p \mid \begin{array}{l} p \text{ a path in } Q \\ \text{of length} \geq 0 \end{array} \right\}$
- (associative) multiplication

$$p \cdot q = \begin{cases} pq & \text{if } \bullet \xrightarrow{p} \bullet \xrightarrow{q} \bullet \text{ is path in } Q \\ 0 & \text{otherwise} \end{cases}$$

Assume now that $\Bbbk$ algebraically closed.

Path algebras are "universal" in following sense:

<u>Theorem</u> (Gabriel, 60s). Let A be any $\Bbbk$-algebra of finite $\Bbbk$-dimension.
Then $\exists$ quiver Q & 2-sided ideal $I \subset \Bbbk Q$
s.t. $\text{Mod-}A \simeq \text{Mod-}(\Bbbk Q / I)$.

<u>Intuition</u>: Quiver path algebras play a role in fin. dim. algebras analogous to what polynomial rings play in affine alg. geom. & commutative algebra.

Reformulation of path algebras without graphs:

For a ring S & S-S-bimodule E, the <u>tensor algebra</u> $T_S(E)$ is:

$$T_S(E) = S \oplus E \oplus (E \otimes_S E) \oplus (E \otimes_S E \otimes_S E) \oplus \cdots$$

with multiplication concatenation of tensors.

Straightforward exercise:

Given a quiver Q with n vertices,

- let $S = \Bbbk Q_0 \simeq \Bbbk^n$ be the subalgebra of $\Bbbk Q$ spanned by length 0 paths,
- let $E = \Bbbk Q_1$ the span of the arrows in $\Bbbk Q$.

Then $\Bbbk Q \simeq T_S(E)$.

More generally if A is any fin. dim. $\Bbbk$-algebra, then A is isomorphic to a quotient of $T_S(E)$ where:

- $S = A / \operatorname{rad} A$
- $E = \operatorname{rad} A / \operatorname{rad}^2 A$

Summary conclusions:

1) tensor algebras are fundamental to study of arbitrary fin. dim. $\Bbbk$-algebras
2) Quiver path algebras are fundamental examples of tensor algebras.

III. Tensor algebras in (finite) tensor categories

Key observation: Just as algebras can be defined in an arbitrary tensor category $\mathcal{C}$, so can modules & bimodules over algebras.

$\Longrightarrow$ Tensor algebras can be defined in $\mathcal{C}$.

For example, a (graded) Hopf action of H on $\Bbbk Q$ is equivalent to a certain kind of tensor algebra in $\mathcal{C} = \mathrm{rep}(H)$.

Broad goal of a research program

Extend the toolset of quivers, which has been highly successful in studying algebras & their modules in $\mathcal{C} = \mathrm{vec}$, to more general tensor categories $\mathcal{C}$.

This line of investigation was initiated in joint work with Etingof & Walton.
Some types of problems which arise:

1. Classify indecomposable semisimple (or exact) algebras S in given $\mathcal{C}$.
2. Fixing S, classify indecomposable bimodules over S in $\mathcal{C}$.

Together, these determine "building blocks" of tensor algebras in $\mathcal{C}$.

Tools like categorical Morita equivalence can be very helpful, e.g. in [EKW] we get results for $\mathcal{C} = \mathrm{rep}(H_8)$ via $\mathcal{C}' = \mathrm{vec}_{D_8}^{\omega}$

Kac-Paljutkin algebra

③ Extend theory of quivers, path algebras & representations to more general $\mathcal{C}$.
(Recent work by Elias-Heng on this)

④ Apply to determine structure and properties of (non-semisimple) algebras in tensor categories, e.g. problems about
- resolutions of modules
- homological dimension
- distinguished representatives of Morita equivalence classes
- Deformations of algebras & modules