Differential Graded Vertex Algebras

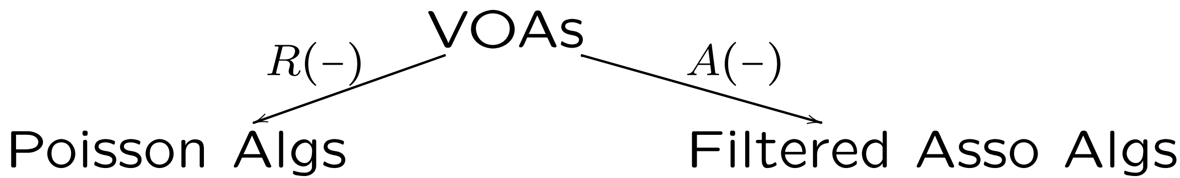
Zongzhu Lin
Kansas State University

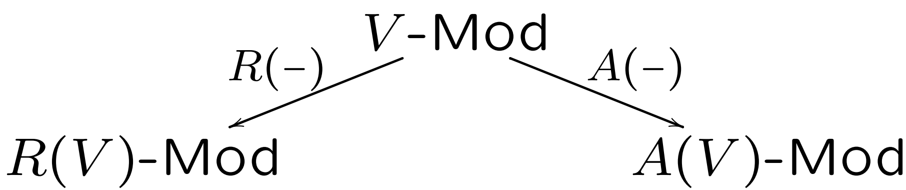
Quantum Symmetries Conferences (QuaSy-Con II)

Notre Dame University

May 24-26, 2024

0. Motivation





Questions:

(1) Compare these module categories.

(2) What are the invariants of these categories?

(3) what are the relevant representation categories in each case?

• $R(V)$ is a commutation and thus there is an associated Poisson variety (scheme), which is could be very singular. Tate provided a process to get a commutative differential graded and graded commutative algebra $\mathcal{T}(R(V))$ by attaching symmetric algebras in each degree. A special case is the Koszul complex (which is just one step) for smooth points. $H^*(\mathcal{T}(R(V))) \cong R(V)$. This is exactly how a dg-scheme is resolved by resolving dg-algebras.

• The Zhu algebra $A(V)$ has more structures than just an algebra, it has all the properties of algebra of differential operators with a filtration so that the associated graded algebra is always a Poisson algebra, which places the role of function algebra of the cotangent bundle.

• We also want to know what derived version of vertex operator algebras are so that their associated $R(V)$ and $A(V)$ should have the derived geometric interpretation.

• The Chiral de Rham complex Ω_X^{ch} over a smooth algebraic variety by Malikov-Schechtman-Vaintrob, which is a complex of sheaves of D-modules with Chiral differential d^{ch}. For each Zariski open set U, $\Omega_X^{ch}(U)$ is a (graded) vertex (super) algebra.

•. Lie algebra (co)homology and the Chevalley-Eilenberg complex seems to appear in our computation of the Yoneda algebra.

1. Associative algebras attached to a VOA

1.0 Vertex algebra

Definition 1. A **vertex operator algebra** $(V, Y(\cdot, x), \mathbf{1}, \omega)$:

V— vector space $/\mathbb{C}$;

$$\begin{array}{rcl} Y(.,x): \; V & \longrightarrow & \mathrm{End}(V)[[z,z^{-1}]] \\ v & \longmapsto & Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \end{array}$$

$\mathbf{1}, \omega \in V$ satisfying $Y(u,z)v \in V((z))$, and

$Y(\mathbf{1},z)v = v$, $\lim_{z\to 0} Y(v,z)\mathbf{1} = v$, + Jacobi identity.

$Y(\omega,z) = \sum_n L_n z^{-n-2}$ makes V a module of the Virasoro Lie algebra with L_0 acts semisimply of integral weights.

Infinitely many products: $(u,v) \mapsto u_n(v)$ for each $n \in \mathbb{Z}$.

1.1. Zhu algebra Given a VOA $(V, Y, \mathbf{1}, \omega)$, there is an associated algebra $A(V) = V/(V \circ V)$

$$a \circ b = \mathrm{Res}_z \left(\frac{(1+z)^{\mathrm{wt}(a)}}{z^2} Y(a,z)b \right).$$

and multiplication on $A(V)$ is induced by

$$a * b = \mathrm{Res}_z \left(\frac{(1+z)^{\mathrm{wt}(a)}}{z} Y(a,z)b \right).$$

$A(V)$ is a filtered associative algebra $F^p A(V) = \overline{\sum_{n \le p} V_n}$

$$F^p A(V) * F^q A(V) \subseteq F^{p+q} A(V).$$

Facts: (1) $V \mapsto A(V)$ is a functor

(2) $A(V)$ is almost commutative in the sense

$$[F^p A(V), F^q A(V)] \subseteq F^{p+q-1} A(V)$$

(3) $\mathrm{gr} A(V) = \oplus_p \mathrm{gr} A(V)_p = \oplus_p F^p A(V)/F^{p-1} A(V)$ is a graded Poisson algebra with Poisson bracket

$$\{\cdot,\cdot\} : \mathrm{gr} A(V)_p \otimes \mathrm{gr} A(V)_q \to \mathrm{gr} A(V)_{p+q-1}$$

defined by the standard Lie bracket in $A(V)$.

If V is a finitely generated CFT type, then $\mathrm{gr} A(V)$ defines a conical Poisson variety $\mathrm{spec}(\mathrm{gr} A(V))$, which has a singularity at the vertex $\{0\}$.

Example 1. Let $V = V_k(\mathfrak{g})$ be the universal affine vertex operator algebra of level k for a finite dimensioal simple Lie algebra $\mathfrak{g}$.

$A(V) = U(\mathfrak{g})$ is the universal enveloping algebra.

$F^p A(V) = \sum_{i \leq p} \mathfrak{g}^i \subseteq U(\mathfrak{g})$.

$\mathrm{gr} A(V) = \mathrm{Sym}^\bullet(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$—the coordinate algebra of $\mathfrak{g}^*$.

The Poisson variety is the affine variety $\mathfrak{g}^*$.

Theorem 1. *If $f : V \to W$ is a surjective vertex operator algebra homomorphism, then the induced maps $A(V) \to A(W)$ and $\mathrm{gr}A(V) \to \mathrm{gr}A(W)$ are surjective as well. Thus the associated Poisson variety for W is a closed conical Poisson subvariety of that for V.*

In particular, for affine VOA as a quotient of $V_{\mathfrak{g}}(k, 0)$, the associated Posson variety is a closed conical Poisson subvariety of $\mathfrak{g}^*$.

Problem: Describe the singularity of $\mathbf{Spec}(\mathrm{gr}A(V))$ at $\{0\}$ in terms of representations of V.

1.2. C_2-algebra $R(V)$

$C_2(V) = \sum_{a \in V} a_{-2}(V)$
$R(V) = V/C_2(V)$ is a commutative associative algebra with a Poisson structure

$$\overline{a} \cdot \overline{b} = \overline{a_{-1}b} \quad \text{and} \quad \{\overline{a}, \overline{b}\} = \overline{a_0 b} \quad \text{for } a, b \in V.$$

Proposition 1. (Arakawa-Lam-Yamada) *There is natural surjective Poisson algebra homomorphism:*

$$\eta_V : R(V) \longrightarrow \mathrm{gr}A(V)$$

Thus inducing $\mathbf{spec}(\mathrm{gr}A(V)) \subseteq \mathbf{spec}(R(V))$

Conjecture: $\mathbf{spec}(\mathrm{gr}A(V))^{red} = \mathbf{spec}(R(V))^{red}$

2. dg vertex algebras

Let $\mathcal{C}h$ be the category of differential (cochain) complexes of $\mathbb{C}$-vector spaces.

- Obj: $(V^{[*]}, d^{[*]})$, Morphisms: chain maps.

- Tensor product: $(V^{[*]} \otimes U^{[*]})^{[n]} = \oplus_{i+j=n} V^{[i]} \otimes U^{[j]}$
- Differential $d^{[*]}_{V \otimes U}(v \otimes u) = d^{[*]}_V(v) \otimes u + (-1)^{|v|} v \otimes d^{[*]}_U(u)$
- Braiding: $T_{V^{[*]}, U^{[*]}}(v \otimes u) = (-1)^{|v||u|} u \otimes v$.

- Internal hom: $\mathcal{H}om^{[p]}(V^{[*]}, U^{[*]}) = \oplus_n \mathrm{Hom}_{\mathbb{C}}(V^{[n]}, U^{[n+p]})$
- Differential : $d^{[*]}(f)(v) = d^{[*]}_U(f(v)) - (-1)^{|f|} f(d^{[*]}_V(v))$

Loop complexes and interpretation in $\mathcal{C}h$

- $\mathbb{C}[t, t^{-1}]$ is graded vector space with $|t| = 2N$, thus a differential complex (concentrated in even degrees)
- $V^{[*]} \otimes \mathbb{C}[t, t^{-1}]$ is a complex. Thus

$$V^{[*]} \otimes t^n = V^{[*]}[-2nN], \qquad V^{[*]} \otimes \mathbb{C}[t, t^{-1}] = \bigoplus_{n \in \mathbb{Z}} V^{[*]}[-2nN].$$

- For any complex $(W^{[*]}, d_W^{[*]})$, we get a complex

$$\begin{aligned}
\mathcal{H}om^{[*]}(V^{[*]} \otimes \mathbb{C}[t, t^{-1}], W^{[*]}) &= \prod_{n \in \mathbb{Z}} \mathcal{H}om^{[*]}(V^{[*]}[-2nN], W^{[*]}) \\
&= \prod_{n \in \mathbb{Z}} \mathcal{H}om^{[*]}(V^{[*]}, W^{[*]}[2nN]) \\
&= \mathcal{H}om^{[*]}(V^{[*]}, W^{[*]})[[x, x^{-1}]].
\end{aligned}$$

In particular, if $W^{[*]} = V^{[*]}$, using the tensor-Hom duality, we get

$$\mathcal{H}om^{[*]}(\mathbb{C}[t,t^{-1}],\mathcal{E}nd^{[*]}(V^{[*]})) = \mathcal{E}nd^{[*]}(V^{[*]})[[x,x^{-1}]]. \quad (1)$$

Here x is regarded as to have degree $-2N$ and $V^{[*]} \otimes x^n = V^{[*]}[2nN]$.

$$\begin{aligned}
&\mathcal{H}om^{[*]}(V^{[*]} \otimes \mathbb{C}[t,t^{-1}], W^{[*]}) \\
&\qquad = \mathcal{H}om^{[*]}(\mathbb{C}[t,t^{-1}], \mathcal{H}om^{[*]}(V^{[*]}, W^{[*]})) \\
&\qquad = \prod_{n\in\mathbb{Z}} \mathcal{H}om^{[*]}(\mathbb{C}[-2nN], \mathcal{H}om^{[*]}(V^{[*]}, W^{[*]})) \\
&\qquad = \prod_{n\in\mathbb{Z}} \mathcal{H}om^{[*]}(V^{[*]}, W^{[*]}))[2nN]
\end{aligned}$$

Definition 2. A **vertex dg-algebra** in $\mathcal{C}h$ is a cochain complex $(V^{[*]}, d_V^{[*]})$ over $\mathbb{C}$ equipped with a chain map (vertex operator map) in $\mathcal{C}h$

$$\begin{aligned} Y(.,x) : V^{[*]} &\longrightarrow \mathcal{H}om^{[*]}(\mathbb{C}[t,t^{-1}], \mathcal{E}nd^{[*]}(V^{[*]})) \\ &= \mathcal{E}nd^{[*]}(V^{[*]})[[x,x^{-1}]] \\ v &\longmapsto Y(v,x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1} \end{aligned}$$

with x of degree $-2N$, and a particular vector $\mathbf{1} \in V^{[0]}$ with $d^{[0]}(\mathbf{1}) = 0$, the vacuum vector, satisfying the following conditions:

- For any $u, v \in V^{[*]}$, $u_n v = 0$ for n sufficiently large, i.e. $Y(u,x)v \in V^{[*]}((x))$ (Truncation property),

- $Y(\mathbf{1}, x) = 1$ (1 is the identity operator on V) (Vacuum property),

- $Y(v, x)\mathbf{1} \in V[[x]]$ and $\lim_{x \to 0} Y(v, x)\mathbf{1} = v$ (Creation property),

- The Jacobi identity

$$x_2^{-1}\delta(\frac{x_1 - x_0}{x_2})Y(Y(u, x_0)v, x_2)$$
$$= x_0^{-1}\delta(\frac{x_1 - x_2}{x_0})Y(u, x_1)Y(v, x_2)$$
$$- (-1)^{|u||v|}x_0^{-1}\delta(\frac{x_2 - x_1}{-x_0})Y(v, x_2)Y(u, x_1),$$

Remarks

• For each $n \in \mathbb{Z}$, the multiplication $V^{[*]} \otimes V^{[*]} \to V^{[*]}[-2N(n+1)]$ defined by $v \otimes u \mapsto v_n(u)$ makes $V^{[*]}$ into a (homologically) graded algebra with a product of degree $-2N(n+1)$ and d_V is a derivation of this algebra.

• For any homogenous $u \in V^{[*]}$ with $d_V(u) = 0$, the linear map $\mathcal{D}_n^u : V^{[*]} \to V^{[*]}[|u| - 2N(n+1)]$ defined by $\mathcal{D}_n^u(v) = v_n(u)$ is chain map.

• The map $u \mapsto \mathcal{D}_n^u$ defines a chain map

$$V^{[*]} \to \mathcal{E}nd^{[*]}(V^{[*]})[-2N(n+1)].$$

- $Y(d_V^{[p]}(v),x)u = d_V^{[p+m]}Y(v,x)u - (-1)^p Y(v,x)d_V^{[m]}(u)$.

- weak associativity:

$$(x_0+x_2)^k Y(Y(u,x_0)v,x_2)w = (x_0+x_2)^k Y(u,x_0+x_2)Y(v,x_2)w.$$

- Weak commutativity:

$$(x_1-x_2)^k \Big(Y(u,x_1)Y(v,x_2) - (-1)^{|u||v|}Y(v,x_2)Y(u,x_1)\Big) = 0.$$

-

$$\sum_{i\geq 0}(-1)^i\binom{l}{i}(u_{m+l-i}v_{n+i} - (-1)^{|v||u|+l}v_{n+l-i}u_{m+i}) = \sum_{i\geq 0}\binom{m}{i}(u_{l+i}v)_{m+n-i}$$

•

$$x_2^{-1}\delta(\frac{x_1 - x_0}{x_2})Y(Y(u,x_0)v,x_2)$$
$$= (-1)^{|u||v|}x_1^{-1}\delta(\frac{x_2 + x_0}{x_1})Y(Y(u,-x_0)v,x_1).$$

Lemma 1. *The vertex dg-algebra is equipped with a cocycle* $\mathcal{D} \in Z^{[2N]}\mathcal{E}nd^{[*]}(V^{[*]})$ *defined by* $\mathcal{D}(v) = v_{-2}\mathbf{1}$ *satisfying* $d_V \circ \mathcal{D} = \mathcal{D} \circ d_V$.

$$[\mathcal{D}, Y(v,x)]^s = Y(\mathcal{D}(v),x) = \frac{d}{dx}Y(v,x).$$

A vertex dg-algebra homomorphism $f : (V^{[*]}, d_V, Y, \mathbf{1}) \to (V'^{[*]}, d_{V'}, Y', \mathbf{1}')$ is a chain map $f \in \mathcal{C}h((V^{[*]}, d_V), (V'^{[*]}, d_{V'}))$

such that

$$f(Y(v,x)u) = Y'(f(v),x)(f(u)) \quad \text{and} \quad f(\mathbf{1}) = \mathbf{1}'.$$

• Given any complex $(V^{[*]}, d_V^{[*]})$, let $H^{[*]}(V^{[*]})$ be the cohomology, which can be regarded as a complex with zero differential.

Theorem 2. *If $(V^{[*]}, d_V^{[*]}, Y, \mathbf{1})$ is a vertex dg-algebra , then there is an induced map $H(Y)(\cdot, x) : H^{[*]}(V^{[*]}) \to \mathcal{E}nd^{[*]}(H^{[*]}(V^{[*]}))[[x, x^{-1}]]$ defining a vertex dg-algebra structure on $H^{[*]}(V^{[*]})$, with $\mathbf{1}$ being the image of $\mathbf{1}$ in $H^{[0]}(V^{[*]})$ since $d_V \mathbf{1} = 0$.*

conformal structure: A vertex operator dg-algebra is a vertex algebra $(V, Y, \mathbf{1})$ together with an element $\omega \in V^{[4N]}$

with $d_V^{[4N]}(\omega) = 0$, called conformal vector (or Virasoro element) such that $Y(\omega, x) = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2}$ (thus $|L(n)| = -2nN$)

- $[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c_V$ for $m, n \in \mathbb{Z}$ (the Virasoro relations) where $c_V \in \mathbb{C}$ is the central charge of V. Thus $\mathfrak{Vir}$ is a graded Lie algebra.
- The linear operator $L(0) \in \mathrm{End}_{\mathbb{C}}^{[0]}(V^{[*]})$ is semisimple and $V^{[*]} = \bigoplus_{n\in\mathbb{Z}} V_n^{[*]}$ with $L(0)v = nv = \mathrm{wt}(v)v$ for $n \in \mathbb{Z}$, $v \in V_n^{[*]}$.
- Furthermore $\omega \in V_2^{[*]}$, $\dim V_n^{[*]} < \infty$, and $V_n^{[*]} = 0$ for $n << 0$.
- $L(-1) = \mathcal{D}$.

- A vertex operator dg-algebra $V^{[*]} = \bigoplus_{n\in\mathbb{Z}} V_n^{[*]}$ is automatically a $\mathbb{Z}$-graded vector space by *conformal* weights.

- We call $\mathrm{wt}(v)$ the *conformal weight* and $|v|$ the *cohomological degree*.

3. Vertex dg-modules

Definition 3. Let $(V^{[*]}, d_V, Y, \mathbf{1})$ be a vertex algebra in $\mathcal{C}h$. A dg-module over a vertex dg-algebra is an object $(M^{[*]}, d_M^{[*]})$ in $\mathcal{C}h$ equipped with a chain map

$$\begin{aligned} Y_M(\cdot, x) : \quad V^{[*]} &\longrightarrow \mathcal{E}nd^{[*]}(M^{[*]})[[x, x^{-1}]] \\ v &\longmapsto Y_M(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \end{aligned}$$

such that for any $u, v \in V$, the following properties are verified :

- For any $u \in V^{[*]}$, $w \in M^{[*]}$, $u_n w = 0$ for n sufficiently large, i.e. $Y_M(u,x)w \in M^{[*]}((x))$. (Truncation property)
- $Y_M(\mathbf{1},x) = \mathrm{Id}_{|M}$. (Vacuum property)
- The Jacobi identity

$$x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)Y_M(Y(u,x_0)v,x_2) =$$
$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y_M(u,x_1)Y_M(v,x_2) -$$
$$(-1)^{|v||u|}x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)Y_M(v,x_2)Y_M(u,x_1).$$

- Morphisms are chain maps.

- **Remark:** a vertex dg-module is automatically the graded module for the graded (super) vertex algebra $(V^{[*]}, Y, \mathbf{1})$ together with a differential of degree 1 and square zero.

• Modules for a vertex operator dg-algebra also has a compatible graded Virasoro Lie algebra module

• Here there is no requirement of whether L_0 is diagonalizable or even locally finite. However, the differential grading defines a graded module structure and thus also a filtered module structure, in addition to the graded/filtered modules structures on weak modules.

• All interesting module categories are abelian and finite (or directed limit).

4. C_2-algebras

Set $C_2^{[*]}(V^{[*]}) = \text{Span}\{u_{-2}v \mid u, v \in V^{[*]}\}$, the **C$_2$-dg-algebra** is the quotient $R^{[*]}(V^{[*]}) = V^{[*]}/C_2^{[*]}(V^{[*]})$. As above, we will write $\overline{u}$ for the class of $u \in V^{[*]}$ in $R^{[*]}(V^{[*]})$.

Theorem 3. *The space $R^{[*]}(V^{[*]})$ is a dg-Poisson algebra with a graded commutative product $\overline{u}\cdot\overline{v} = \overline{u_{-1}v}$ and a Poisson bracket $\{\overline{u}, \overline{v}\}_{R(V)} = \overline{u_0 v}$. If $V^{[*]}$ is a vertex operator dg-algebra, then $R^{[*]}(V^{[*]})$ is a weight-graded dg-algebra*

$$R^{[*]}(V^{[*]}) = \bigoplus_{(p,n)\in\mathbb{Z}^2} R^{[p]}(V^{[*]})_n.$$

with $R^{[p]}(V^{[*]})_n$ the image of $V_n^{[p]}$ in $R^{[*]}(V^{[*]})$.

Let $M^{[*]}$ be a dg-module for a vertex dg-algebra $V^{[*]}$. Define $C_2^{[*]}(M^{[*]}) = \mathrm{Span}\{v_{-2}m \mid v \in V^{[*]}, m \in M^{[*]}\}$ and set $R_V^{[*]}(M^{[*]}) = M^{[*]}/C_2^{[*]}(M^{[*]})$.

Theorem 4. *If $M^{[*]}$ is dg-module for a vertex dg-algebra, then $R^{[*]}(M^{[*]})$ is a dg-Poisson module for $R^{[*]}(V^{[*]})$. If $M^{[*]} \in V\text{-Mod}^{gr}$ for a vertex operator dg-algebra $V^{[*]}$, then $R^{[*]}(M^{[*]})$ is also a $\mathbb{Z}^2$-graded module for $R^{[*]}(V^{[*]})$.*

6. Zhu algebras

Let $V^{[*]}$ be a vertex operator dg-algebra. There are two filtered structures on $V^{[*]}$.

$$(V^{[*]}, F^\bullet, d) = (\cdots \subseteq F^p V^{[*]} \subseteq F^{p+1} V^{[*]} \subseteq \cdots \subseteq V^{[*]})$$

with $F^pV^{[*]} = \oplus_{i\leq p}V^{[i]}$ and The other is a weight filtration

$$(V^{[*]}, W_\bullet, d) = (\cdots \subseteq W_nV^{[*]} \subseteq W_{n+1}V^{[*]} \subseteq \cdots \subseteq V^{[*]})$$

with $W_nV^{[*]} = \oplus_{m\leq n}V_m^{[*]}$.

The Zhu algebra is $A(V^{[*]}) = V^{[*]}/O(V^{[*]})$, is the same Zhu algebra of the vertex operator algebra forgetting the differential, and cohomological grading. Now the differential gradation and weight gradations are no longer graded.

- $A(V^{[*]})$ is a differential filtered algebra with the ascending filtration $(F^pA(V^{[*]}))_{p\in\mathbb{Z}}$ where $F^pA(V^{[*]})$ is the image of $\bigoplus_{q\leq p}V_*^{[q]}$ in $A(V^{[*]})$.

• The image of the conformal vector ω is in the center of $A(V^{[*]})$ for the super commutative bracket $[\cdot,\cdot]^s$, i.e., $[\omega] * [v] = [v] * [\omega]$ for all $[v] \in A(V^{[*]})$.
• $A(V^{[*]})$ has a weight filtration $(W_n A(V^{[*]}))_{n\in\mathbb{Z}}$ where $W_n A(V^{[*]})$ is the image of $\bigoplus_{i\le n} V_i^{[*]}$ in the $A(V^{[*]})$.

We define

$$\{\widetilde{x},\widetilde{y}\}_{F^\bullet V} = [u] * [v] - (-1)^{|u||v|}[v] * [u] \text{ mod } F^{|x|+|y|-3}A(V^{[*]}).$$

The map

$$\{\cdot,\cdot\}_{F^\bullet V} : \mathrm{gr}^{[p]} A(V^{[*]}) \otimes \mathrm{gr}^{[q]} A(V^{[*]}) \longrightarrow \mathrm{gr}^{[p+q-2]} A(V^{[*]})$$

is well-defined.

Corollary 1. $\mathrm{gr}^{[*]} A(V^{[*]})$ *is a graded commutative dg-Poisson algebra in the category* $\mathbf{grVec}$. *The product is induced by* $*$, *the Poisson bracket is* $\{.,.\}_{F^\bullet V}$ *and of degree* -2, *and the differential is induced by* $d_V^{[*]}$. *In particular, for* $\widetilde{x}, \widetilde{y} \in \mathrm{gr}^{[*]} A(V^{[*]})$ *dg-homogeneous with respective preimages* $u \in V^{[|\widetilde{x}|]}$, $v \in V^{[|\widetilde{y}|]}$, *we have*

$$\widetilde{x} * \widetilde{y} = [\widetilde{u_{-1}v}] \qquad \textit{and} \qquad \{\widetilde{x}, \widetilde{y}\}_{F^\bullet V} = [\widetilde{u_0 v}].$$

However, the weight filtration does now get a dg algebras structure on the associated graded algebra $\mathrm{gr}_* A(V^{[*]}) = \bigoplus_{n\in\mathbb{Z}} W_n A(V^{[*]})/W_{n-1} A(V^{[*]})$.

Proposition 2. *The algebra* $\operatorname{gr}_* A(V^{[*]})$ *is a weight-graded, differential filtered algebra. Furthermore, the product is differential filtered commutative.*

Proposition 3. *The algebra* $\operatorname{gr}_* \operatorname{gr}^{[*]} A(V^{[*]}) \cong \operatorname{gr}^{[*]} \operatorname{gr}_* A(V^{[*]})$ *is a weight graded dg-Poisson algebra. The Poisson bracket is of degree* -2 *for the cohomological degree, and of degree* -1 *for the weight. Furthermore, the dg-grading makes it a graded commutative algebra.*

7. Maps from C_2-algebras to associated graded Zhu algebras

We define the following map:

$$\begin{array}{rccc} \eta_{F^\bullet V}: & R^{[*]}(V^{[*]}) & \longrightarrow & \mathrm{gr}^{[*]} A(V^{[*]}) \\ & u + C_2^{[|u|]}(V^{[*]}) & \longmapsto & u + O(V^{[*]}) + \bigoplus_{p<|u|} V^{[p]}. \end{array}$$

Proposition 4. *The map $\eta_{F^\bullet V}$ is a surjective morphism of dg-Poisson algebras.*

$$\begin{array}{rccc} \eta_{W_\bullet V}: & R^{[*]}(V^{[*]}) & \longrightarrow & \mathrm{gr}_* A(V^{[*]}) \\ & u + C_2^{[*]}(V^{[*]})_{\mathrm{wt}(u)} & \longmapsto & u + O(V^{[*]}) + \bigoplus_{n<\mathrm{wt}(u)} V_n^{[*]}. \end{array}$$

The map $\eta_{W_\bullet V}$ is well-defined.

Proposition 5. *The map* $\eta_{W_\bullet V}$ *is a surjective morphism of weight-graded, differential filtered, differential filtered commutative algebras.*

$$\begin{array}{rccc} \eta_{F^\bullet W_\bullet V}: & R^{[*]}(V^{[*]}) & \longrightarrow & \operatorname{gr}_* \operatorname{gr}^{[*]} A(V^{[*]}) \\ & u + C_2^{[p]}(V^{[*]})_n & \longmapsto & u + O(V^{[*]}) + \bigoplus_{\substack{q \leq p \\ m < n}} V_m^{[q]} + \bigoplus_{\substack{q < p \\ m \leq n}} V_m^{[q} \end{array}$$

Proposition 6. *The map* $\eta_{F^\bullet W_\bullet V}$ *is a surjective morphism of weight-graded dg-Poisson algebras.*

• More on comparing the module categories of these unexpected objects: $\mathrm{gr}_*(V^{[*]})$, $\mathrm{gr}^{[*]}A(V^{[*]})$, and $\mathrm{gr}_* \mathrm{gr}^{[*]}A(V^{[*]})$. These are done and the diagram is complicated to draw here.

• They are commutative in a certain sense with additional structures. We don't have a good geometric interpretation yet.

• The cohomological properties have not yet been explored. For example, the cocycles are also vertex subalgebras, and how one defines the homotopy vertex algebra, or derived vertex algebras where the Jacobi identity holds only up to a homotopy equivalence.

8. Examples of dg vertex algebras

• Given a dg LA $\mathfrak{g}$ with a supersymmetric bilinear form $\langle -, - \rangle$ (which is a cocycle). Then one can define affine dg vertex algebras. $V_{\mathfrak{g}}(\ell, 0)$

• dg vertex Lie algebra (also the dg conformal algebra defines a universal vertex algebras,

• Joyce constructed vertex algebra structure on the cohomology groups of a class of quotient stacks including quiver representation stacks. We want to lift this vertex algebra structure on the standard resolution complexes so that the cohomology vertex algebra would be the Joyce vertex algebra. The special case is when the quiver is a point, then Kontsevivh-Soibelman COHA should have a dg vertex algebra with dg lift.

- $\mathcal{W}$-algebra BRST construction automatically gives dg-vertex algebra structure ($N = 0$).

- Resolving vertex algebra in terms of smooth vertex algebras.

THANK YOU!