Inverting Hamiltonian Reduction

An Introduction

Christopher Raymond

Université Laval

joint w/ David Ridout and Justine Fasquel

About Me:

- Mathematical physicist interested in 2D Conformal Field Theories and their related mathematics

  VOA-Modules / $\otimes$-Categories / Number Theory / String Theory / TQFT

- Algebra and Rep theory Background.

  Goal: Construct and classify representations relevant to conformal field theory.

- Particular interest in "logarithmic" CFTs (logCFTs)

  - CFTs where the relevant rep theory is non-semisimple.

  - Often involve things called "relaxed" highest weight modules.

A very loose algebraists view of Chiral CFT:

What is a 2D Conformal field theory?

2D Quantum field theory with conformal symmetry.

Quantum states organised into modules over the symmetry algebra.

Symmetry algebra

Vertex Operator Algebra. $\longrightarrow$

State space

Collection of VOA-modules

$\downarrow$

n-point functions

- Tensor structure $\boxtimes$
- Duals
- Modular properties
- Fusion Rules

$\longleftarrow$

Chiral CFT

Nice: Modular tensor Category

Less nice: Possibly non-rigid
Definitely non-semisimple
analogue of MTC.

A first Example:

A VOA is a ($\mathbb{Z}$-graded) vector space V, w/ a map $Y: V \longrightarrow \mathrm{End}(V)[\![z^{\pm 1}]\!]$

$$Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-\Delta} \; ; \quad v \in V_{(\Delta)}, \Delta \in \mathbb{Z}, \quad v_n \in \mathrm{End}(V)$$

The product is on fields (OPE):

$$Y(v,z)\,Y(u,w) = \sum_{n\in\mathbb{Z}} Y(v_{-\Delta+n}u, w)\,(z-w)^{-n}$$

Example: $L_k(\mathfrak{sl}_2)$. Generated by $e(z), h(z), f(z)$ (all $\Delta = 1$), $k \in \mathbb{C}$, satisfying:

$$h(z)e(w) \sim \frac{2e(w)}{z-w}, \qquad h(z)f(w) \sim -\frac{2f(w)}{z-w}, \qquad h(z)h(w) \sim \frac{2k}{(z-w)^2}, \qquad e(z)f(w) \sim \frac{k}{(z-w)^2} + \frac{h(w)}{z-w}$$

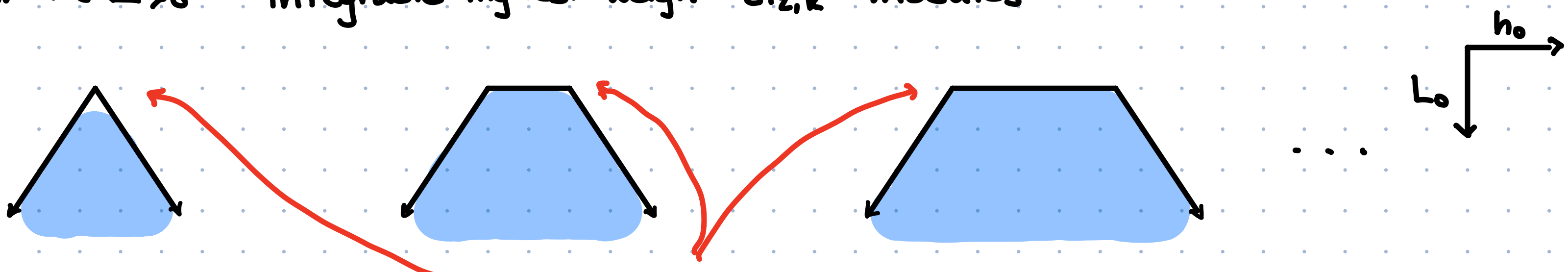
The endomorphisms $\{e_n, h_n, f_n, \mathbb{1} \mid n \in \mathbb{Z}\}$ span an $\widehat{\mathfrak{sl}}_{2,k}$ affine Lie algebra

$$L_k(\mathfrak{sl}_2)\text{-modules} \longleftrightarrow \widehat{\mathfrak{sl}}_{2,k}\text{-modules}$$

*Under some technical assumptions.

Which modules play a role?

$L_k(sl_2)$ for $k \in \mathbb{Z}_{>0}$: Integrable highest weight $\widehat{sl}_{2,k}$-modules.



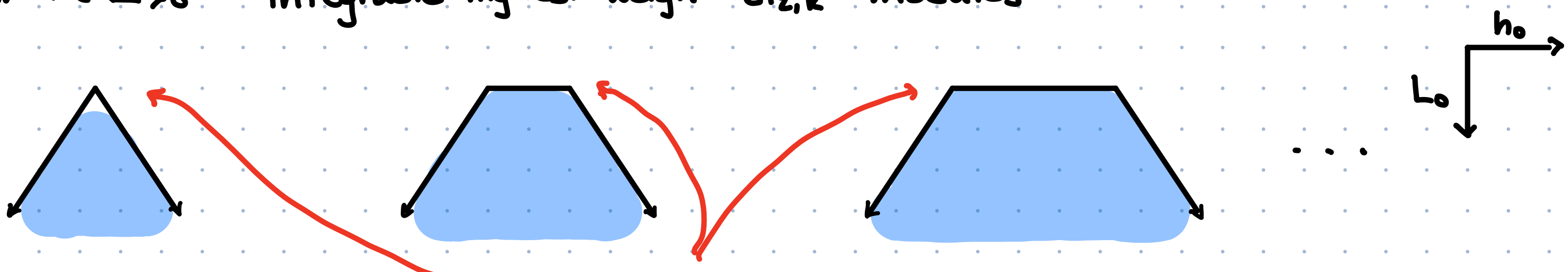
$\mathbb{Z}$-graded modules. Top space is fin. dim. sl_2-irrep

Finitely-many highest weight irreducible modules + complete reducibility

$\Rightarrow$ SL_2 WZW models, Nice modular tensor categories. A little bit boring...

Which modules play a role?

$L_k(sl_2)$ for $k \in \mathbb{Z}_{\geq 0}$: Integrable highest weight $\widehat{sl}_{2,k}$-modules.



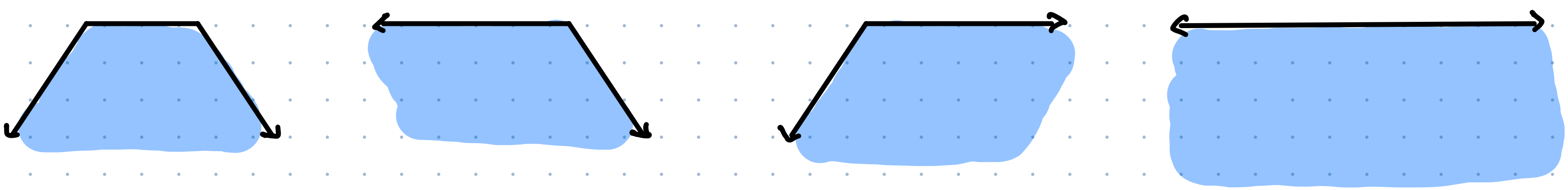
$\mathbb{Z}$-graded modules. Top space is fin. dim. sl_2-irrep

Finitely-many highest weight irreducible modules + complete reducibility

$\Rightarrow$ SL_2 WZW models, Nice modular tensor categories. A little bit boring...

$L_k(sl_2)$ for $k+2 = \frac{u}{v}$, $u \in \mathbb{Z}_{\geq 3}$, $v \in \mathbb{Z}_{\geq 2}$, $\gcd(u,v) = 1$

"Relaxed h.w. modules"



Top space: fin. dim. | Verma | Conj. Verma | Dense

Inf. many irreducibles, hard to classify relaxed modules, but necessary

Q: How do we get a better handle on relaxed modules?

Quantum Hamiltonian Reduction :

For $\mathfrak{g}$ a simple Lie algebra, there is always an assoc. $L_k(\mathfrak{g})$

Given a nilpotent $f \in \mathfrak{g}$, we can "reduce" the symmetry of $L_k(\mathfrak{g})$ via QHR

$$L_k(\mathfrak{g}) \longrightarrow H^0_{DS}(L_k(\mathfrak{g})) \cong W_k(\mathfrak{g}, f)$$

$W_k(\mathfrak{g}, f)$ unique up to conj. of $f \in \mathfrak{g}$.

Output is a "W-algebra". Map extends to modules:

When f is principal, i.e. $\begin{bmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & 1 & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}$, and k is admissible [Kac-Wakimoto]

$$L_k(\mathfrak{g})\text{-mod} \longrightarrow W_k(\mathfrak{g}, f)\text{-mod}.$$

$$\text{irred. h.w.} \longmapsto \text{irred. h.w. or } 0.$$

Quantum Hamiltonian Reduction :

For $\mathfrak{g}$ a simple Lie algebra, there is always an assoc. $L_k(\mathfrak{g})$

Given a nilpotent $f \in \mathfrak{g}$, we can "reduce" the symmetry of $L_k(\mathfrak{g})$ via QHR

$$L_k(\mathfrak{g}) \longrightarrow H^0_{DS}(L_k(\mathfrak{g})) \cong W_k(\mathfrak{g}, f)$$

$W_k(\mathfrak{g}, f)$ unique up to conj. of $f \in \mathfrak{g}$.

Output is a "W-algebra". Map extends to modules:

When f is principal, i.e. $\begin{bmatrix} 0 & & 0 \\ 1 & 0 & \\ 0 & \ddots & \\ & 1 & 0 \end{bmatrix}$, and k is admissible [Kac-Wakimoto]

$$L_k(\mathfrak{g})\text{-mod} \longrightarrow W_k(\mathfrak{g}, f)\text{-mod}.$$

$$\text{irred. h.w.} \longmapsto \text{irred. h.w. or } 0.$$

$W_k(\mathfrak{g}, f_{prin})$-mod is semisimple and $W_k(\mathfrak{g}, f_{prin})$ cannot have relaxed modules

Hard to classify $L_k(\mathfrak{sl}_2)$ theories satisfy these conditions

$$\mathfrak{sl}_2 : \ W_k(\mathfrak{sl}_2, f) \cong \text{Virasoro with } c = 1 - \frac{6(u-v)^2}{uv}$$

Non-unitary Minimal Models

These are well understood. Can we lift back to $L_k(\mathfrak{sl}_2)$ relaxed?

Inverting Hamiltonian Reduction:

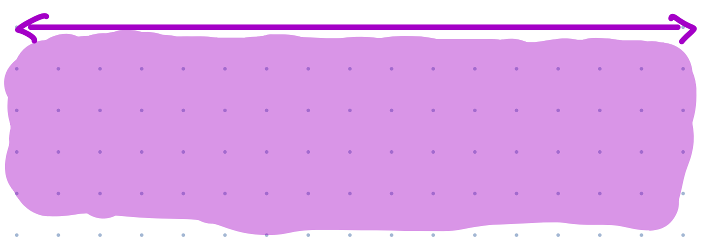
Idea: Can we "reverse" QHR?

[Adamovic, '17] Yes! Only for k admissible and $\notin \mathbb{Z}_{>0}$.

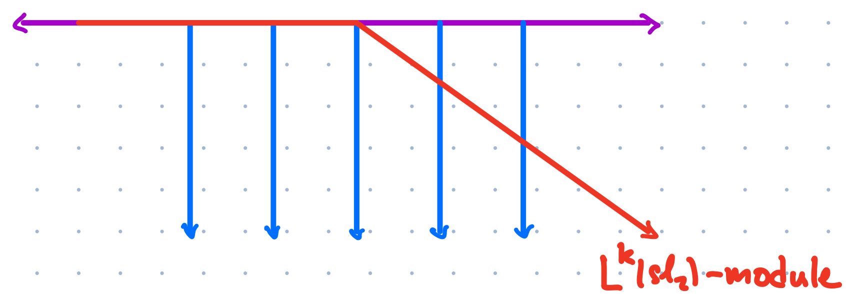
$L_k(sl_2) \hookrightarrow Vir \otimes \Pi$ Inverse Hamiltonian Reduction

Π is a "half-lattice VOA". Π-modules are always relaxed

Π-modules



$Vir \otimes \Pi$-modules



Realises relaxed $L_k(sl_2)$-modules inside $Vir \otimes \Pi$-modules

Using all Vir (finite and well understood) and Π-modules (easy) constructs

<u>all</u> relaxed $L_k(sl_2)$-modules. Irreducible quotients, characters, modular data

all easier to extract.

How far does this extend?

General Idea: Start at $W_k(\mathfrak{g}, f_{prin})$, work back through chains of Hamiltonian Reductions

$$L_k(\mathfrak{sl}_2) \longleftarrow \mathrm{Vir}$$

[Adamovic, '17]

$$L_k(\mathfrak{sl}_3) \longleftarrow BP_k \longleftarrow W_{3,k}$$

[Adamovic-Creutzig-Genra '21] [Adamovic-Kawasetsu-Ridout '20, '23]

[Fehily-Ridout '21]

Used to construct and classify all relaxed modules

Some additional results for: general $\mathfrak{sl}_n$ [Fehily '22, '23].

$\mathfrak{so}_{2n+1}$ [Fasquel-Nakatsuka '23]

$\mathfrak{sp}_4$ [Beem-Meneghelli '21]

super [Adamovic '17]

[Fasquel-R-Ridout '24 (soon!)]

Classification of relaxed $L_k(\mathfrak{sl}_3)$-modules via IQHR for

$$k+3 = \frac{u}{2}, \quad u \geq 3 \ \text{odd}$$

+ Modularity and fusion rules (character).

Summary:

- Inverse Hamiltonian Reduction gives a new way of constructing and classifying Affine and W-algebra modules relevant in log CFT.
- Constructs all relaxed $L_k(\mathfrak{g})$-modules in known cases.
- Obtain all simple modules by degeneration
- New concrete examples of nonsemisimple fusion categories.
- Method naturally gives braid/modular info from Grothendieck Ring
- Can build $L_k(\mathfrak{g})$-modules w/ infinite-dimensional wt. spaces.