Invertible Fusion Categories

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Introduction

Theorem (Etingof, Nikshych, and Ostrik 2011, Thm 3.1)

For $\mathcal{C}, \mathcal{D}$ fusion over $\mathbb{C}$,

$$\left(\quad \mathcal{Z}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{D}) \quad \right) \iff \left(\quad \mathcal{C} \overset{Mor}{\sim} \mathcal{D} \quad \right)$$

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Why are fusion categories so nice?

Morita equivalence

Idea

Just as different bases can give different presentations of the same vector space, different algebras can give different presentations of the same category.

Definition

Two algebras A and B are said to be Morita equivalent and we'll write $A \overset{\text{Mor}}{\sim} B$ whenever

$$A\text{-Mod} \simeq B\text{-Mod}$$

The Following are Equivalent

Morita equivalence

Let A and B be f.d. algebras.

- $A\text{-Mod} \simeq B\text{-Mod}$
- $\exists M, N$ such that $N \otimes_B M \cong A$, and $M \otimes_A N \cong B$
- $\exists$ a progenerator $M \in B\text{-Mod}$ with $A \cong \mathrm{End}_B(M)^{\mathrm{op}}$

Theorem [Morita 1957]

$$A \overset{\text{Mor}}{\sim} B \implies Z(A) \cong Z(B).$$

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The Following are Equivalent

Morita equivalence

Let $\mathcal{C}$ and $\mathcal{D}$ be fusion categories.

- $\mathcal{C}$-Mod $\simeq \mathcal{D}$-Mod
- $\exists \mathcal{M}, \mathcal{N}$ such that $\mathcal{N} \boxtimes_{\mathcal{D}} \mathcal{M} \cong \mathcal{C}$, and $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} \cong \mathcal{D}$
- $\exists$ an separable $\mathcal{M} \in \mathcal{D}$-Mod with $\mathcal{C} \simeq \text{End}_{\mathcal{D}}(\mathcal{M})^{\otimes\text{-op}}$

Theorem

$$\mathcal{C} \overset{\text{Mor}}{\sim} \mathcal{D} \implies \mathcal{Z}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{D}).$$

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Fusion for general $\mathbb{K}$

Definition

A fusion category over a field $\mathbb{K}$ is a $\mathbb{K}$-linear, abelian, finite, semisimple, rigid monoidal category with simple unit object $\mathbb{1}$.

Keep in mind...

- Schur's Lemma: $\text{End}(X)$ is a division algebra
- Eckmann-Hilton: $\text{End}(\mathbb{1})$ is a field!
- Objects interact with $\text{Gal}\big(\text{End}(\mathbb{1})/\mathbb{K}\big)$

Deligne products

- X and Y simple, and yet $X \boxtimes Y$ might not be!
- $\mathrm{End}(X \boxtimes Y) \cong \mathrm{End}(X) \otimes_{\mathbb{K}} \mathrm{End}(Y)$
- These can have projections! ↑

Over $\mathbb{R}$...

$\downarrow A \otimes_{\mathbb{R}} B \rightarrow$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C} \oplus \mathbb{C}$	$M_2(\mathbb{C})$
$\mathbb{H}$	$\mathbb{H}$	$M_2(\mathbb{C})$	$M_4(\mathbb{R})$

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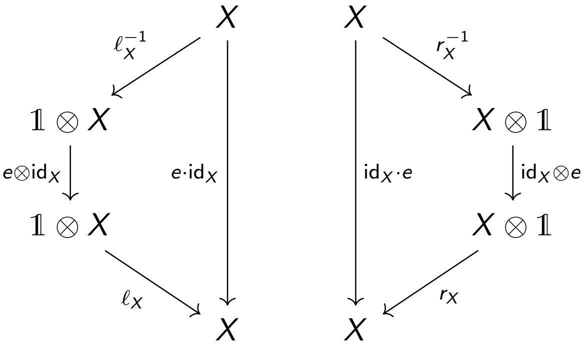
Left/Right embeddings

End($\mathbb{1}$) can behave differently on the left and right.



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$\mathbb{L}$-Bim$_{\mathbb{K}}$

- $\mathbb{L}/\mathbb{K}$ a Galois extension
- $\mathbb{L}$-Bim$_{\mathbb{K}}$: The category of bimodules for $\mathbb{L} \in \mathrm{Vec}_{\mathbb{K}}$
- Simple objects $\mathbb{L}_g$ for $g \in \mathrm{Gal}(\mathbb{L}/\mathbb{K})$
- Morita equivalent to $\mathrm{Vec}_{\mathbb{K}}$

$$\mathbb{L}_g \otimes \mathbb{L}_h = \mathbb{L}_{gh}$$

$$g(\lambda) \cdot \mathrm{id}_{\mathbb{L}_g} = \mathrm{id}_{\mathbb{L}_g} \cdot \lambda$$

A new category $\mathrm{Vec}_{\mathbb{L}}^{\omega}(\mathrm{Gal}(\mathbb{L}/\mathbb{K}))$

- Simple objects $\mathbb{L}_g$ for $g \in \mathrm{Gal}(\mathbb{L}/\mathbb{K})$
- New associator $\omega \in H^3(\mathrm{Gal}(\mathbb{L}/\mathbb{K})\,;\, \mathbb{L}^\times)$
- $\mathcal{Z}(\mathcal{C}) = \mathrm{Vec}_{\mathbb{K}}$
- Morita equivalent to $\mathrm{Vec}_{\mathbb{K}}$?

$$
\begin{array}{ccc}
(\mathbb{L}_a \otimes \mathbb{L}_b) \otimes \mathbb{L}_c & \xrightarrow{\alpha_{\mathbb{L}_a,\mathbb{L}_b,\mathbb{L}_c}} & \mathbb{L}_a \otimes (\mathbb{L}_b \otimes \mathbb{L}_c) \\
\| & & \| \\
\mathbb{L}_{ab} \otimes \mathbb{L}_c & & \mathbb{L}_a \otimes \mathbb{L}_{bc} \\
\| & & \| \\
\mathbb{L}_{abc} & \xrightarrow[\omega(a,b,c)\cdot \mathrm{id}_{\mathbb{L}_{abc}}]{} & \mathbb{L}_{abc}
\end{array}
$$

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Inflation

Given a tower $\mathbb{L}/\mathbb{E}/\mathbb{K}$...

$$\begin{array}{ccc} H^n\left(\mathrm{Gal}(\mathbb{E}/\mathbb{K})\,;\,\mathbb{E}^\times\right) & \longrightarrow & H^n\left(\mathrm{Gal}(\mathbb{E}/\mathbb{K})\,;\,\mathbb{L}^\times\right) \\ \downarrow & & \downarrow \\ H^n\left(\mathrm{Gal}(\mathbb{L}/\mathbb{K})\,;\,\mathbb{E}^\times\right) & \longrightarrow & H^n\left(\mathrm{Gal}(\mathbb{L}/\mathbb{K})\,;\,\mathbb{L}^\times\right) \end{array}$$

$$H^n\left(\mathrm{Gal}(\mathbb{E}/\mathbb{K})\,;\,\mathbb{E}^\times\right) \to H^n\left(\mathrm{Gal}(\mathbb{L}/\mathbb{K})\,;\,\mathbb{L}^\times\right) \to \cdots \to H^n\left(\mathrm{Gal}(\mathbb{K}^{\mathrm{sep}}/\mathbb{K})\,;\,(\mathbb{K}^{\mathrm{sep}})^\times\right)$$

Categorical inflation

- We introduce a new construction $\mathcal{C} \mapsto \mathsf{Infl}_{\mathbb{E}}^{\mathbb{L}}(\mathcal{C})$.
- $\mathsf{Infl}_{\mathbb{E}}^{\mathbb{L}}(\mathcal{C})$ is Morita equivalent to $\mathcal{C}$.

Theorem (S-Snyder)

$$\mathsf{Infl}_{\mathbb{E}}^{\mathbb{L}}\Big(\ \mathsf{Vec}_{\mathbb{E}}^{\omega}(\mathsf{Gal}(\mathbb{E}/\mathbb{K}))\ \Big) \simeq \mathsf{Vec}_{\mathbb{L}}^{\mathsf{infl}_{\mathbb{E}}^{\mathbb{L}}(\omega)}(\mathsf{Gal}(\mathbb{L}/\mathbb{K}))$$

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Categorical inflation (sketch)

When $\text{End}_{\mathcal{C}}(\mathbb{1}) \cong \mathbb{E}$...

$$\mathbb{L}\text{-Bim}_{\mathbb{K}} \overset{\text{Mor}}{\sim} \text{Vec}_{\mathbb{K}}$$

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$$\mathbb{L}\text{-Bim}_{\mathbb{K}} \boxtimes \mathcal{C} \overset{\text{Mor}}{\sim} \text{Vec}_{\mathbb{K}} \boxtimes \mathcal{C} \simeq \mathcal{C}$$

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This isn't fusion! $\rightarrow$ $\mathbb{L}\text{-Bim}_{\mathbb{K}} \boxtimes \mathcal{C} \overset{\text{Mor}}{\sim} \mathrm{Vec}_{\mathbb{K}} \boxtimes \mathcal{C} \simeq \mathcal{C}$

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$$\mathbb{1}_1 \otimes (\mathbb{L}\text{-Bim}_{\mathbb{K}} \boxtimes \mathcal{C}) \otimes \mathbb{1}_1 \overset{\text{Mor}}{\sim} \mathbb{L}\text{-Bim}_{\mathbb{K}} \boxtimes \mathcal{C} \overset{\text{Mor}}{\sim} \text{Vec}_{\mathbb{K}} \boxtimes \mathcal{C} \simeq \mathcal{C}$$

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$$\mathbb{L}\text{-Bim}_{\mathbb{K}} \overset{\text{Mor}}{\sim} \text{Vec}_{\mathbb{K}}$$

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$$\mathbb{1}_1 \otimes (\mathbb{L}\text{-Bim}_{\mathbb{K}} \boxtimes \mathcal{C}) \otimes \mathbb{1}_1 \overset{\text{Mor}}{\sim} \mathbb{L}\text{-Bim}_{\mathbb{K}} \boxtimes \mathcal{C} \overset{\text{Mor}}{\sim} \text{Vec}_{\mathbb{K}} \boxtimes \mathcal{C} \simeq \mathcal{C}$$

$$\mathbb{1}_1 \otimes (\mathbb{L}\text{-Bim}_{\mathbb{K}} \boxtimes \mathcal{C}) \otimes \mathbb{1}_1 =: \text{Infl}_{\mathbb{E}}^{\mathbb{L}}(\mathcal{C})$$

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Absolute Galois cohomology

Some comments on $H^*(\mathbb{K}; \mathbb{G}_m) = H^*\left(\mathrm{Gal}(\mathbb{K}^{\mathrm{sep}}/\mathbb{K}) ; (\mathbb{K}^{\mathrm{sep}})^\times\right)$

- $H^0(\mathbb{K}; \mathbb{G}_m) = \mathbb{K}^\times$,
- $H^1(\mathbb{K}; \mathbb{G}_m) = 0$, (Hilbert Thm 90)
- $H^2(\mathbb{K}; \mathbb{G}_m) \cong \mathrm{Br}(\mathbb{K})$, (Algebras with $Z(A) = \mathbb{K}$, up to Morita eq.)
- $H^3(\mathbb{K}; \mathbb{G}_m) =???$

G-normal algebras and their cocycles

Let $G = \mathrm{Gal}(\mathbb{L}/\mathbb{K})$. An $\mathbb{L}$-central simple algebra D is G-normal if $[D] \in \mathrm{Br}(\mathbb{L})^G$.

Theorem (Teichmüller 1940)

There is a homomorphism $T : \mathrm{Br}(\mathbb{L})^G \to H^3\left(\mathrm{Gal}(\mathbb{L}/\mathbb{K})\,;\,\mathbb{L}^\times\right)$.

Theorem (Eilenberg and Mac Lane 1948)

$$\mathrm{im}(T) = \ker\left(H^3\left(\mathrm{Gal}(\mathbb{L}/\mathbb{K})\,;\,\mathbb{L}^\times\right) \xrightarrow{\mathit{infl}} H^3(\mathbb{K};\mathbb{G}_m) \right).$$

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Results

Theorem (S-Snyder)

$$\left(\mathrm{Vec}_{\mathbb{L}}^{\omega}(\mathrm{Gal}(\mathbb{L}/\mathbb{K})) \overset{Mor}{\sim} \mathrm{Vec}_{\mathbb{K}} \right) \iff \left(\omega = T([D]) \right)$$

Invertible fusion categories

Definition

A fusion category $\mathcal{C}$ over $\mathbb{K}$ is called (Morita) invertible if $\mathcal{Z}(\mathcal{C}) = \mathrm{Vec}_{\mathbb{K}}$.

$\mathrm{Inv}(\mathbb{K}) = \{\text{Invertible fusion categories}/\mathbb{K} \text{ up to } \overset{\text{Mor}}{\sim}\}$ ($\leftarrow$ a group under $\boxtimes$)

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Theorem (S-Snyder)

Every invertible fusion category is Morita equivalent to some $\mathrm{Vec}_{\mathbb{L}}^{\omega}(\mathrm{Gal}(\mathbb{L}/\mathbb{K}))$, *for some* $\mathbb{L}$ *and* ω.

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Results

Theorem (S-Snyder)

Inflation induces an isomorphism

$$H^3(\mathbb{K}; \mathbb{G}_m) \cong \mathrm{Inv}(\mathbb{K}) .$$

Results

Theorem (S-Snyder)

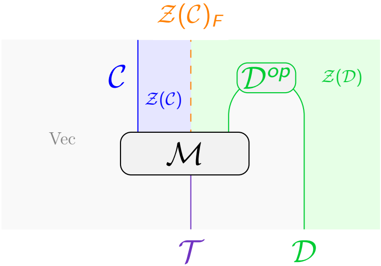
For $\mathcal{C}, \mathcal{D}$ fusion over $\mathbb{K}$, with
$\mathcal{C} : \mathrm{Vec}_{\mathbb{K}} \to \mathcal{Z}(\mathcal{C})$, and $\mathcal{D} : \mathrm{Vec}_{\mathbb{K}} \to \mathcal{Z}(\mathcal{D})$ invertible 1-morphisms in **Br-mFus**$_{\mathbb{K}}$,

$$\left(\mathcal{Z}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{D}) \right) \iff \left(\mathcal{C} \overset{Mor}{\sim} \mathrm{Vec}^{\omega}_{\mathbb{E}}(\mathrm{Gal}(\mathbb{E}/\mathbb{K})) \boxtimes \mathcal{D} \right).$$

The $\mathrm{Vec}^{\omega}_{\mathbb{E}}(\mathrm{Gal}(\mathbb{E}/\mathbb{K}))$ can be removed if $[\omega] = 1 \in H^3(\mathbb{K}; \mathbb{G}_m)$

Proof sketch

- Use $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{D})$ to build $\mathcal{Z}(\mathcal{C})_F$.
- $\mathcal{C} \boxtimes_{\mathcal{Z}(\mathcal{C})} \mathcal{Z}(\mathcal{C})_F \boxtimes_{\mathcal{Z}(\mathcal{D})} \mathcal{D}^{\text{rev}}$ is invertible.
- Compose with $\mathcal{D}^{\text{op}}$.



Finding examples

- $H^3(\mathbb{K}; \mathbb{G}_m) = 0$ for any local field!
- $H^3(\mathbb{K}; \mathbb{G}_m) = 0$ for any global field!
- $H^3(\mathbb{C}(x, y, z); \mathbb{G}_m) \neq 0$ is hard to show!
- (Uematsu 2014) and (Riman 2020) prove this while computing brauer groups of certain varieties.

A spectral sequence

$\mathrm{Inv}(\mathbb{L})^G$	$\cdots$		
$\mathrm{Br}(\mathbb{L})^G$	$H^1(G;\mathrm{Br}(\mathbb{L}))$	$\cdots$	
0	0	0	0
$\mathbb{K}^\times$	0	$H^2(G;\mathbb{L}^\times)$	$H^3(G;\mathbb{L}^\times)$

$d_3 : \mathrm{Br}(\mathbb{L})^G \to H^3(G;\mathbb{L}^\times)$

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Future work

Conjecture

For $\mathcal{C}, \mathcal{D}$ *fusion over* $\mathbb{K}$, *with* $\mathrm{End}(\mathbb{1}_{\mathcal{Z}(\mathcal{C})}) \cong \mathrm{End}(\mathbb{1}_{\mathcal{Z}(\mathcal{D})}) \cong \mathbb{K}$,

$$\left(\quad \mathcal{Z}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{D}) \quad \right) \iff \left(\quad \mathcal{C} \overset{Mor}{\sim} \mathrm{Vec}_{\mathbb{E}}^{\omega}(\mathrm{Gal}(\mathbb{E}/\mathbb{K})) \boxtimes \mathcal{D} \quad \right).$$

The $\mathrm{Vec}_{\mathbb{E}}^{\omega}(\mathrm{Gal}(\mathbb{E}/\mathbb{K}))$ *can be removed if* $[\omega] = 1 \in H^3(\mathbb{K}; \mathbb{G}_m)$

Fusion 2-categories?

- $H^4(\mathbb{K}; \mathbb{G}_m)$ is NOT the group of invertible fusion 2-categories!
- The Witt group of [Davydov et al. 2011] comes into the picture.
- A Galois theory of categorified fields in the style of [Johnson-Freyd 2017] will be necessary.

Thank You!

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