Comodule Algebras over H_8

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Outline

Module Categories

Let $\Bbbk$ be an algebraically closed field and H a finite dimensional Hopf algebra.

Module Categories

A left module category $\mathcal{M}$ over $\mathrm{Rep}(H)$ is an abelian category with an exact bifunctor $\overline{\otimes} : \mathrm{Rep}(H) \times \mathcal{M} \to \mathcal{M}$ with natural associativity and unit isomorphisms:

$$m_{X,Y,M} : (X \otimes_{\Bbbk} Y)\overline{\otimes}M \to X\overline{\otimes}(Y\overline{\otimes}M), \quad \ell_M : \mathbb{1}\overline{\otimes}M \to M$$

$X, Y \in \mathrm{Rep}(H)$, $M \in \mathcal{M}$ satisfying some compatibility conditions.

Module Categories

Module Functors

A $\mathrm{Rep}(H)$-module functor $F : \mathcal{M} \to \mathcal{N}$ is a functor with natural isomorphism:

$$s_{X,M} : F(X \overline{\otimes} M) \to X \overline{\otimes} F(M), \quad X \in \mathrm{Rep}(H), M \in \mathcal{M}.$$

Two $\mathrm{Rep}(H)$-module categories are equivalent if there exists a $\mathrm{Rep}(H)$-module functor realizing the equivalence.

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Two $\text{Rep}(H)$-module categories are equivalent if there exists a $\text{Rep}(H)$-module functor realizing the equivalence.

Goal

Given a finite dimensional Hopf algebra, find all exact indecomposable $\text{Rep}(H)$-module categories (up to equivalence).

Comodule Algebras

Let H be a Hopf algebra and $\otimes = \otimes_{\mathbb{k}}$,

H-Comodule Algebra

A $\mathbb{k}$-vector space A is an H-comodule algebra if:

1. A is a $\mathbb{k}$-algebra,
2. A is an H-comodule via $\lambda : A \to H \otimes A$,
3. λ is a morphism of $\mathbb{k}$-algebras.

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We say A is right H-simple if there are no nontrivial right ideals $J \subseteq A$ such that $\lambda(J) \subseteq H \otimes J$.

The space of coinvariants is $A^{coH} = \{a \in A : \lambda(a) = 1 \otimes a\}$.

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Notice $1_A \in A^{coH}$.

We say A has **trivial coinvariants** if $\dim(A^{coH}) = 1$.

Left A-modules

Let $\Bbbk$ be an algebraically closed field of characteristic zero and H a finite dimensional Hopf algebra.

Andruskiewitsch and Mombelli (2007)

If $\mathcal{M}$ is an exact indecomposable module category over $\text{Rep}(H)$ then $\mathcal{M} \simeq {}_A\mathcal{M}$ for some right H-simple left comodule algebra A with trivial coinvariants.

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$\text{Rep}(H)$-module structure on ${}_A\mathcal{M}$

Note that $X \otimes M$ is a left $H \otimes A$ module for $X \in \text{Rep}(H)$ and $M \in {}_A\mathcal{M}$.

Thus λ makes $X \otimes M$ a left A-module.

Explicitly:

$$a \cdot (x \otimes m) = \sum (a_{(-1)} \cdot x) \otimes (a_{(0)} \cdot m)$$

where $\lambda(a) = \sum a_{(-1)} \otimes a_{(0)}$.

Morita Equivalence

Morita Equivariant Equivalence

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If A, B are Morita equivariant equivalent, then A, B are Morita equivalent as $\mathbb{k}$-algebras.

Goal

Given a finite dimensional Hopf algebra, determine all right H-simple left comodule algebras with trivial coinvariants up to Morita equivariant equivalence.

Past Results

Mombelli was able to classify exact indecomposable module categories for pointed Hopf algebras in the following cases:

1. $G(H) \cong \mathbb{Z}_n$ (2009),
    1. Taft algebras,
    2. Small quantum group $u_q(\mathfrak{sl}_2)$,

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Note

This process depends greatly on the structure of H (or more specifically, the structure of the associated coradically graded Hopf algebra).

Outline

The Kac-Paljutkin Hopf Algebra

Let $\Bbbk$ be algebraically closed and characteristic zero.

H_8

The Kac-Paljutkin Hopf algebra H_8 is generated by x, y, z with x, y grouplike, generating a Hopf subalgebra isomorphic to $\Bbbk[K]$ and z satsifying:

$$z^2 = \frac{1}{2}(1 + x + y - xy) \qquad xz = zy \qquad yz = zx$$

and

$$\Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z).$$

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1. H_8 is the unique 8 dimensional semisimple non-commutative, non-cocommutative Hopf algebra.
2. H_8 is self dual, cosemisimple, and not pointed

H_8-Comodules

Cosemisimplicity

The following is a list of all simple H_8-comodules:

1. The one dimensional comodules V_g where $\lambda(V_g) = g \otimes V_g$ for $g \in G(H) = \{1, x, y, xy\}$.
2. The two dimensional comodule V_2 generated by v, w with

$$\lambda(v) = \frac{1}{2}\Big[yz \otimes (v + w) + z \otimes (v - w)\Big]$$
$$\lambda(w) = \frac{1}{2}\Big[xyz \otimes (v + w) - xz \otimes (v - w)\Big]$$

Every comodule algebra A is a direct sum of copies of these comodules.

Group Comodule Algebras

Let A be a right H_8-simple H_8-comodule algebra with trivial coinvariants.

In a similar way that $\Bbbk[K] \subseteq H_8$, there exists a subcomodule subalgebra $A_K \subseteq A$ such that A_K is a right simple $\Bbbk[K]$-comodule algebra with trivial coinvariants.

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A_K

If (A, λ) is a right H_8-simple comodule algebra with trivial coinvariants then A_K is isomorphic to $\mathbb{k}^\psi[F]$ for some subgroup $F \subseteq K$ and 2-cocycle $\psi \in Z^2(F, \mathbb{k}^\times)$.

In particular, A_K is isomorphic to one of:

1. $\mathbb{k}$,
2. $\mathbb{k}[C_2]$,
3. $\mathbb{k}[K]$,
4. $\mathbb{k}^\psi[K]$ ($\cong M_2(\mathbb{k})$).

Extending to H_8

Extending A_K

We now need to classify the algebras A such that $A \neq A_K$.

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Consider the basis $\{v_i, w_i\}_{i=1}^{n_2}$ for $V_2^{\oplus n_2}$.

$$\lambda(v_i) = \frac{1}{2}\Big[yz \otimes (v_i + w_i) + z \otimes (v_i - w_i)\Big]$$
$$\lambda(w_i) = \frac{1}{2}\Big[xyz \otimes (v_i + w_i) - xz \otimes (v_i - w_i)\Big]$$

Notice

$$\begin{aligned} 4\lambda(v_i v_j) = 1 &\otimes (v_i v_j - v_i w_j - w_i v_j - w_i w_j) \\ &+ x \otimes (v_i v_j + v_i w_j - w_i v_j + w_i w_j) \\ &+ y \otimes (v_i v_j - v_i w_j + w_i v_j + w_i w_j) \\ &+ xy \otimes (v_i v_j + v_i w_j + w_i v_j - w_i w_j). \end{aligned}$$

So $v_i v_j \in A_K$.

Extending to H_8 (cont)

Let e_g generate V_g for $g \in \{1, x, y, xy\}$, there exists scalars $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij}$ such that:

$$v_i v_j - v_i w_j - w_i v_j - w_i w_j = 4\alpha_{ij},$$
$$v_i v_j + v_i w_j - w_i v_j + w_i w_j = 4\beta_{ij} e_x,$$
$$v_i v_j - v_i w_j + w_i v_j + w_i w_j = 4\gamma_{ij} e_y,$$
$$v_i v_j + v_i w_j + w_i v_j - w_i w_j = 4\delta_{ij} e_{xy},$$

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v_i v_j + v_i w_j + w_i v_j - w_i w_j &= 4\delta_{ij} e_{xy},
\end{aligned}
$$

This system has a unique solution:

$$
\begin{aligned}
v_i v_j &= \alpha_{ij} + \beta_{ij} e_x + \gamma_{ij} e_y + \delta_{ij} e_{xy}, \\
v_i w_j &= -\alpha_{ij} + \beta_{ij} e_x - \gamma_{ij} e_y + \delta_{ij} e_{xy}, \\
w_i v_j &= -\alpha_{ij} - \beta_{ij} e_x + \gamma_{ij} e_y + \delta_{ij} e_{xy}, \\
w_i w_j &= -\alpha_{ij} + \beta_{ij} e_x + \gamma_{ij} e_y - \delta_{ij} e_{xy}.
\end{aligned}
$$

A list of H_8 comodule algebras with trivial coinvariants

The following is an exhaustive list of isomorphism classes of right H_8-simple comodule algebras with trivial coinvariants:

1. $\Bbbk$,
2. The subalgebras of H_8 isomorphic to $\Bbbk[C_2]$ with generator x, y or xy.
3. The subalgebra $\Bbbk[K]$,
4. The twisted group algebra $\Bbbk^{\psi}[K]$ ($\cong M_2(\Bbbk)$),

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5. The algebra $A_{xy}^{q} \cong \mathbb{k}^4$, ($A_{xy}^{q} \cong V_1 \oplus V_{xy} \oplus V_2$ as comodules)
6. (H_8, Δ).

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Morita Equivalence

As $\mathbb{k}$-algebras, the following are Morita equivalent:

1. The algebras $\mathbb{k}$ and $\mathbb{k}^{\psi}[K]$,
2. The three algebras isomorphic to $\mathbb{k}[C_2]$,
3. The algebras A^q_{xy} and $\mathbb{k}[K]$.

Morita Equivariant Equivalence

$\Bbbk$ and $M_2(\Bbbk)$

The algebras $\Bbbk$ and $\Bbbk^{\psi}[K]$ are Morita equivariant equivalent.

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The algebras generated by x and y are Morita equivariant equivalent but are not Morita equivariant equivalent to the algebra generated by xy.

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$\mathbb{k}[K]$ and A^q_{xy}

The algebras $\mathbb{k}[K]$ and A^q_{xy} are not Morita equivariant equivalent.

Morita Equivariant Inequivalent Algebras

The following is an exhaustive list of Morita equivariant inequivalent right H_8-simple H_8-comodule algebras with trivial coinvariants.

1. $\mathbb{k}$,
2. The subalgebra of H_8 generated by x,
3. The subalgebra of H_8 generated by xy,
4. The subalgebra $\mathbb{k}[K]$ generated by x and y,
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Rep(H_8)-module categories

If $\mathcal{M}$ is an exact indecomposable Rep(H_8)-module category then $\mathcal{M} \simeq {}_A\mathcal{M}$ for exactly one A in the list above.

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Remark

Each of the algebras in the above list is isomorphic to a coideal subalgebra of H_8.

Outline

Remarks

Rep(H_8)-module categories

Etingof, Kinser and Walton classified Rep(H_8)-module categories using H_8-module algebras and a categorical Morita equivalence

$$C(D_8, \omega, C_2, 1) \overset{\otimes}{\sim} \mathrm{Rep}(H_8).$$

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$\text{Rep}(H_8)$-module categories

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Classification of finite dimensional Hopf algebras with coradical H_8 was done by Yuxing Shi in 2016.

Thank You/Questions

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