Math 20580 Tutorial Week 01 - Aug 27 2015 Name ____________________

Section ________________

Tutorial Worksheet

1. Find all solutions to the linear system by following the given steps

$$\begin{cases} x + 2y + 3z = 2 \\ 2x + 3y + z = 4 \\ y + z = 8 \end{cases}.$$

Step 1. Write down the argumented matrix (coefficient and constants) of the system.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ \underline{2} & \underline{3} & \underline{1} & \underline{4} \\ \underline{0} & \underline{1} & \underline{1} & \underline{8} \end{array}\right]$$

Step 2. Replace the second row of the matrix with the second row subtracted by 2 times the first row. In our notation, we would write $R_2 \mapsto R_2 - 2R_1$ to denote this operation. This gives us the matrix

$$\xrightarrow{R_2 \mapsto R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & \underline{-1} & \underline{-5} & \underline{0} \\ \underline{0} & \underline{1} & \underline{1} & \underline{8} \end{array}\right]$$

Step 3. Keep performing the row operations until you can solve the equation.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & -5 & 0 \\ 0 & 1 & 1 & 8 \end{array}\right] \xrightarrow{R_2 \mapsto -R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 5 & 0 \\ 0 & 1 & 1 & 8 \end{array}\right] \xrightarrow{R_1 \mapsto R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 2 \\ 0 & 1 & 5 & 0 \\ 0 & 1 & 1 & 8 \end{array}\right]$$

$$\xrightarrow{R_3 \mapsto R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 2 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & -4 & 8 \end{array}\right] \xrightarrow{R_3 \mapsto \frac{R_3}{-4}} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 2 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_2 \mapsto R_2 - 5R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & 2 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_1 \mapsto R_1 + 7R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -12 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & -2 \end{array}\right] \implies \begin{cases} x = -12 \\ y = 10 \\ z = -2 \end{cases}.$$

2. As a generalization to the first question, can you find the solution to the system

$$\begin{cases} x + 2y + 3z = a \\ 2x + 3y + z = b \\ y + z = c \end{cases}$$

for arbitrary real numbers a, b, and c?

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & -1 & -5 & b-2a \\ 0 & 1 & 1 & c \end{array}\right] \xrightarrow{R_2 \mapsto -R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & 5 & 2a-b \\ 0 & 1 & 1 & c \end{array}\right] \xrightarrow{R_1 \mapsto R_1 - 2R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & -3a+2b \\ 0 & 1 & 5 & 2a-b \\ 0 & 1 & 1 & c \end{array}\right] \xrightarrow{R_3 \mapsto R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -7 & -3a+2b \\ 0 & 1 & 5 & 2a-b \\ 0 & 0 & -4 & c-2a+b \end{array}\right]$$

$$\xrightarrow{R_3 \mapsto \frac{R_3}{-4}} \left[\begin{array}{ccc|c} 1 & 0 & -7 & -3a+2b \\ 0 & 1 & 5 & 2a-b \\ 0 & 0 & 1 & \frac{-c+2a-b}{4} \end{array}\right] \xrightarrow{R_2 \mapsto R_2 - 5R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & -3a+2b \\ 0 & 1 & 0 & -\frac{a}{2}+\frac{b}{4}+\frac{5c}{4} \\ 0 & 0 & 1 & \frac{-c+2a-b}{4} \end{array}\right] \xrightarrow{R_1 \mapsto R_1 + 7R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{a}{2}+\frac{b}{4}-\frac{7c}{4} \\ 0 & 1 & 0 & -\frac{a}{2}+\frac{b}{4}+\frac{5c}{4} \\ 0 & 0 & 1 & \frac{-c+2a-b}{4} \end{array}\right]$$

$$\Longrightarrow \begin{cases} x = \frac{a}{2}+\frac{b}{4}-\frac{7c}{4} \\ y = -\frac{a}{2}+\frac{b}{4}+\frac{5c}{4} \\ z = \frac{-c+2a-b}{4} \end{cases}.$$

3. Which matrices below are in echelon form? Which are in reduced echelon form?

$$A = \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Echelon form : $\underline{A, B, C, D}$.

Reduced echelon form : $\underline{A, C}$.

4. Which columns of the matrix below are pivot and which are free?

$$\begin{bmatrix} 1 & 2 & -1 & -3 & 2 \\ 2 & 5 & -1 & -6 & 6 \\ 3 & 7 & -2 & -8 & 8 \end{bmatrix}$$

Remark: A *free column* is a column which corresponds to a free variable.

Sol: To find pivot columns of a matrix, the first step is to reduce the matrix into *row reduced echelon form*.

$$\begin{bmatrix} 1 & 2 & -1 & -3 & 2 \\ 2 & 5 & -1 & -6 & 6 \\ 3 & 7 & -2 & -8 & 8 \end{bmatrix} \xrightarrow{R_2 \mapsto R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -1 & -3 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 3 & 7 & -2 & -8 & 8 \end{bmatrix} \xrightarrow{R_3 \mapsto R_3 - 3R_1}$$

$$\begin{bmatrix} 1 & 2 & -1 & -3 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \mapsto R_3 - R_2} \begin{bmatrix} 1 & 2 & -1 & -3 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \mapsto R_1 + 3R_3}$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \mapsto R_1 - 2R_2} \begin{bmatrix} \color{red}{1} & \color{red}{0} & -3 & \color{red}{0} & -2 \\ \color{red}{0} & \color{red}{1} & 1 & \color{red}{0} & 2 \\ \color{red}{0} & \color{red}{0} & 0 & \color{red}{1} & 0 \end{bmatrix}.$$

From the above row reduced echelon form, we know that the columns marked in red are pivot, and other columns are free.

5. Find all solutions to the linear system

$$\begin{cases} x & - & 3y & + & z & - & w & = & 2 \\ 2x & - & 6y & + & 3z & - & w & = & 3 \\ 3x & - & 9y & + & 5z & - & w & = & 4 \end{cases}.$$

Sol. We suggest everyone solve this linear system via matrix method. i.e., find the row reduced echelon form of the corresponding matrix of this linear system.

$$\left[\begin{array}{cccc|c} 1 & -3 & 1 & -1 & 2 \\ 2 & -6 & 3 & -1 & 3 \\ 3 & -9 & 5 & -1 & 4 \end{array}\right] \xrightarrow{R_2 \mapsto R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & -3 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 3 & -9 & 5 & -1 & 4 \end{array}\right] \xrightarrow{R_3 \mapsto R_3 - 3R_1}$$

$$\left[\begin{array}{cccc|c} 1 & -3 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & 2 & -2 \end{array}\right] \xrightarrow{R_3 \mapsto R_3 - R_2} \left[\begin{array}{cccc|c} 1 & -3 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \mapsto R_1 - R_2}$$

$$\left[\begin{array}{cccc|c} 1 & -3 & 0 & -2 & 3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Longleftrightarrow \begin{cases} x - 3y - 2w & = 3 \\ z + w & = -1 \end{cases}.$$

We can see from the row reduced matrix that the pivot columns correspond to the variable x and z. So we could choose x and z as basic variables, and y, w as free variables. In the other words, we can regard y and w as parameters which help us to represent x and z. Therefore, we can write our solution of this linear system as

$$\begin{cases} x & = 3y + 2w + 3 \\ y & = y \\ z & = -1 - w \\ w & = w \end{cases}.$$

Remark: This linear system has infinity many solutions, so you might have a different solution from above. In order to make sure your solution is correct, it has to match the following two conditions:

(1) 2 free variable. i.e., 2 parameters.

(2) satisfies the linear system.