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Sequential Accuracy in Parameter Estimation for Population Correlation Coefficients

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Abstract

Correlation coefficients are effect size measures that are widely used in psychology and related disciplines for quantifying the degree of relationship of two variables, where different correlation coefficients are used to describe different types of relationships for different types of data. We develop methods for constructing a sufficiently narrow confidence interval for 3 different population correlation coefficients with a specified upper bound on the confidence interval width (e.g., .10 units) at a specified level of confidence (e.g., 95%). In particular, we develop methods for Pearson's *r*, Kendall's tau, and Spearman's rho. Our methods solve an important problem because existing methods of study design for correlation coefficients generally require the use of supposed but typically unknowable population values as input parameters. We develop sequential estimation procedures and prove their desirable properties in order to obtain sufficiently narrow confidence interval for population correlation coefficients without using supposed values of population parameters, doing so in a distribution-free environment. In sequential estimation procedures, supposed values of population parameters for purposes of sample size planning are not needed, but instead stopping rules are developed and once satisfied, they provide a rule-based stop to the sampling of additional units. In particular, data in sequential estimation procedures are collected in stages, whereby at each stage the estimated population values are updated and the stopping rule evaluated. Correspondingly, the final sample size required to obtain a sufficiently narrow confidence interval is not known a priori, but is based on the outcome of the study. Additionally, we extend our methods to the squared multiple correlation coefficient under the assumption of multivariate normality. We demonstrate the effectiveness of our sequential procedure using a Monte Carlo simulation study. We provide freely available R code to implement the methods in the MBESS package.

Translational Abstract

We develop methods for constructing sufficiently narrow confidence intervals for population correlation coefficients with a specified upper bound on the confidence interval width (e.g., .10 units) at a specified level of confidence (e.g., 95%). This is an important contribution because wide confidence intervals convey that there is much uncertainty associated with the estimate. Narrow confidence intervals, however, convey that at some level of confidence a population parameter value is contained within a narrow range. Traditional methods for planning sample size, such as from a power analysis or accuracy in parameter estimation perspective required population parameters or supposed values as input, where often these values are unknown. Using the sequential methods that we develop, the most difficult part of planning sample size for narrow confidence intervals has been eliminated. We demonstrate the effectiveness of our sequential procedure using a Monte Carlo simulation study. Additionally, we provide a freely available R code to implement the methods in the MBESS package.

Keywords: correlation coefficient, sample size planning, accuracy in parameter estimation (AIPE), sequential analysis, research design

Correlation coefficients provide scale-free measures of the magnitude, direction, and strength of the linear relationship between two variables and lies in the interval $[-1, 1]$. The Pearson's product-moment correlation coefficient is often used to assess the degree of linear relationship when two variables are quantitative. However, other correlation coefficients exist when variables are

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ordinal, such as the Kendall's tau or Spearman's rho rank correlation coefficients. In this article, we develop sequential methods for obtaining accurate estimates of selected population correlation coefficients. We begin with the Pearson's product-moment correlation due to its popularity in psychology and related fields before generalizing to Kendall's tau and Spearman's rho, all without distributional assumptions. Additionally, we extend the methods to the squared multiple correlation coefficient in the multiple regression framework under the assumption of multivariate normality. Our work builds on Kelley, Darku, and Chattopadhyay (2018), but is distinct in important ways, as we will discuss.

Using Pearson's product-moment correlation coefficient as an example, suppose $\{(X_1, Y_1), \ldots, (X_n, Y_n)\}$ is a random sample from a bivariate distribution of arbitrary form, F, with covariance σ_{XY} and with the marginal distributions of X and Y having population variances σ_X^2 and σ_Y^2, respectively. Throughout the article we assume that observations are drawn from a homogeneous population. The population Pearson's product-moment correlation coefficient of X and Y is given by

$$\rho = \frac{\sigma_{XY}}{\sqrt{\sigma_X^2 \sigma_Y^2}}. \tag{1}$$

In psychology and related disciplines, the Pearson's product-moment correlation coefficient is often a primary outcome variable of interest. For this reason, many authors have heavily invested in methodological work for estimation of and inference for the population Pearson's product-moment correlation coefficient in an effort to better describe quantitative relationships, plan studies that will estimate the correlation coefficient, and perform inferential procedures for the Pearson's product-moment correlation coefficient. For example, Wolf and Cornell (1986) and Bonett and Wright (2000) emphasized the importance of estimating the population correlation coefficient with a narrow confidence interval, specifically under the assumption of a bivariate normal distribution. Under the same distribution assumptions, Moinester and Gottfried (2014) provided a review of several methods for constructing a narrow confidence interval for the population correlation coefficient. Holding constant the population of interest, the effect size of interest, any bias of the estimator, and the confidence interval coverage, a narrower confidence interval for the parameter is preferred to a wider confidence interval because it illustrates more precision of the estimated value of the parameter of interest. Holding constant or decreasing any bias, one way of increasing precision, and thereby improving accuracy, is to increase the sample size (e.g., Kelley & Maxwell, 2003; Maxwell, Kelley, & Rausch, 2008).

The existing approaches of planning sample size for obtaining a narrow confidence interval for the population correlation coefficient are based on supposed values of one or more population parameters in the context of bivariate normal distributions (e.g., Bonett & Wright, 2000; Corty & Corty, 2011; Moinester & Gottfried, 2014). A framework of sample size planning known as accuracy in parameter estimation (AIPE), which has been developed for constructing sufficiently narrow confidence intervals for a variety of population effect sizes, has traditionally been based on supposed population parameter values (e.g., Kelley, 2007c, 2008; Kelley & Lai, 2011; Kelley & Maxwell, 2003; Kelley & Rausch, 2006; Lai & Kelley, 2011a, 2011b; Pornprasertmanit & Schneider, 2014; Terry & Kelley, 2012). However, a potential problem is the requirement of one or more supposed values of the population parameters, which will generally be unknown.[1] In general, applications of AIPE and power analysis also depend on supposed population values. When using supposed population values, that is, treating a supposed value as if it were the true population value, the obtained sample size estimates can differ dramatically from what the theoretically optimal sample size value would be if the population parameters were known. In such situations, even small differences in the supposed and actual value of a parameter can lead to large differences in the planned versus (actually) required sample size.

Although, Fisher's (1915) z-transform method can be used to find the confidence interval for the population correlation coefficient, it is based on the assumption of bivariate normality. However, in our method for the Pearson product–moment correlation coefficient, we work in a distribution-free scenario, as our confidence interval procedure is built upon the asymptotic distribution of sample correlation coefficient proposed by Lee (1990). Unlike Bonett and Wright (2000), Corty and Corty (2011), and Moinester and Gottfried (2014), our approach is more flexible because (a) it does not require the assumption of the bivariate normal distribution of the two variables and (b) supposed values of the population parameters are not needed to plan the sample size. These two points are critical.

We use a *sequential approach* to find a narrow confidence interval for the population Pearson's product-moment correlation coefficient, which we call *sequential AIPE*. This approach is similar to the "fixed-width confidence interval" method, in which the width of the confidence interval is prespecified. Sequential AIPE differs from the fixed-width confidence interval approach because sequential AIPE aims to find the minimum value of the sample size such that the confidence interval is sufficiently narrow by prespecifying the upper bound on the confidence interval width (e.g., Mukhopadhyay & Chattopadhyay, 2012; Mukhopadhyay & De Silva, 2009; Sproule, 1985). In particular, the fixed-width confidence interval procedure deals with construction of a confidence interval for a population parameter that has a width which is exactly equal to the prespecified value of the confidence interval width. By contrast, in sequential AIPE, the aim is to obtain a sufficiently narrow confidence interval for a population parameter such that the confidence interval is not wider than the prespecified width.

Under the distribution-free scenario, the exact sampling distribution of the sample correlation coefficient cannot be obtained. This is because, in the distribution-free environment, no underlying distribution is assumed, such as a bivariate normal. Unlike Fisher's method, our method is developed under the distribution-free scenario, where we use the asymptotic distribution of the sample correlation coefficient developed by Lee (1990) to obtain a

[1] In the context of statistical power, an alternative approach that does not require the specification of the population parameter(s) is to specify the minimally important effect size of interest, which bases statistical power on the minimum parameter value of interest that would be practically of interest or theoretically interesting. O'Brien and Castelloe (2007) discuss that this approach has potential problems, because for important outcomes in which any non-zero effect is important, the planned sample sizes can be extremely large.

sufficiently narrow confidence interval for the population Pearson product–moment correlation coefficient, ρ, using the smallest possible sample size.

We first discuss the (traditional) AIPE for the Pearson product–moment correlation coefficient and propose a sequential estimation procedure, which extends the ideas of Kelley et al. (2018). We then extend the methods used for the Pearson product–moment correlation coefficient to Kendall's tau rank correlation coefficient and Spearman's rank correlation coefficient. The methods of Kelley et al. (2018) were for a generalized effect size consisting of the ratio of linear functions. Thus, the methods that we develop here are for three types of correlation coefficients, which represents a fundamentally different type of effect size than that given in Kelley et al. (2018). Nevertheless, in both cases, our methods use a sequential procedure for constructing a sufficiently narrow confidence interval (i.e., no larger than the specified width) with a specified level of confidence without requiring supposed population values. Importantly, we make all of these developments in a distribution-free environment. The distribution-free environment is important because there is often no assurance that the underlying distribution of the data for which the correlation coefficient will be calculated would be known (e.g., bivariate normal, bivariate gamma). Using R code via the MBESS package (Kelley, 2007b, 2017), we provide an example that demonstrates how the method can be used in practice for the Pearson's product-moment correlation coefficient. Finally, we provide an extension of the sequential procedure in order to obtain a sufficiently narrow confidence interval for the squared multiple correlation coefficient, yet here we assume multivariate normality rather than working in a distribution-free environment (due to the current limitations in the distribution-free literature for confidence intervals for the population squared multiple correlation coefficient). For all of the different correlation coefficients discussed, we provide the results of Monte Carlo simulations that illustrate the characteristics of the procedures in a variety of scenarios.

Accuracy in Parameter Estimation of Pearson's Product Moment Correlation Coefficient

Pearson's product-moment correlation coefficient continues to serve an important role in psychology and related disciplines. The sample correlation coefficient based on n observations is given by

$$r_n = \frac{S_{XYn}}{\sqrt{S_{Xn}^2 S_{Yn}^2}}. \tag{2}$$

where S_{XY_n} is the sample covariance, and S_{Xn}^2 and S_{Yn}^2 are respectively the sample variances corresponding to X and Y. The expressions of S_{XYn}, S_{Xn}^2, and S_{Yn}^2 are given in Equations 64–67 in Appendix A. Using Lee (1990), the asymptotic variance of r_n is ξ^2/n, where

$$\xi^2 = \frac{\rho^2}{4}\left(\frac{\mu_{40}}{\sigma_X^4} + \frac{\mu_{04}}{\sigma_Y^4} + \frac{2\mu_{22}}{\sigma_X^2\sigma_Y^2} + \frac{4\mu_{22}}{\sigma_{XY}^2} - \frac{4\mu_{31}}{\sigma_{XY}\sigma_X^2} - \frac{4\mu_{13}}{\sigma_{XY}\sigma_Y^2}\right) \tag{3}$$

and $\mu_{ij} = \mathrm{E}[(X-\mu_X)^i(Y-\mu_Y)^j]$ is the $(i, j)^{\text{th}}$ joint central moment of X and Y. Then the approximate $(1 - \alpha)100\%$ confidence interval for ρ is given by

$$\left[r_n - z_{\alpha/2}\frac{\xi}{\sqrt{n}}, r_n + z_{\alpha/2}\frac{\xi}{\sqrt{n}}\right], \tag{4}$$

where $z_{\alpha/2}$ is the $(1 - \alpha/2)^{\text{th}}$ quantile of the standard normal distribution. The width of the confidence interval defined in Equation 4 is given by

$$w_n = 2z_{\alpha/2}\frac{\xi}{\sqrt{n}}. \tag{5}$$

In AIPE problems, the sample size required to achieve the sufficient accuracy is solved by specifying the upper bound on the width of the confidence interval, ω. So, for a given ω, we have

$$2z_{\alpha/2}\frac{\xi}{\sqrt{n}} \le \omega, \tag{6}$$

which implies that the necessary sample size to construct $(1 - \alpha)100\%$ confidence interval for ρ will be

$$n \ge \left\lceil \frac{4z_{\alpha/2}^2\xi^2}{\omega^2} \right\rceil \equiv n_\omega, \tag{7}$$

where $\lceil x \rceil$ is the ceiling function which is the smallest integer greater than or equal to x (e.g., $\lceil 49.2 \rceil = 50$). Here, n_ω is the theoretically optimal sample size required to make the $(1 - \alpha)100\%$ confidence interval for ρ sufficiently narrow provided that the asymptotic variance, ξ^2, is known. The optimal sample size, n_ω, is unknown generally as in reality ξ^2 is unknown. We note that the supposed values of ξ^2 cannot be used to estimate n_ω as this may not guarantee that the condition in Equation 7 is satisfied. We use a sequential procedure, which does not need a supposed population parameter value to preplan the sample size required that will satisfy the condition given in Equation 7. Here, we use a consistent estimator of ξ^2 in our sequential procedure to estimate the optimal sample size. The consistent estimator of ξ^2 is given by

$$\hat{\xi}_n^2 = \max\{V_n^2, n^{-\gamma}\}, \gamma > 0, \tag{8}$$

where V_n^2 is given in Appendix A. We note that the estimator V_n^2 is a moment-based estimator of ξ^2 (in Equation 70). There is a positive chance, even though negligible, that V_n^2 may come out to be negative in some situations. In order to avoid that scenario, if it arises, we use the term $n^{-\gamma}$. Any choice of γ will not affect the consistency property of $\hat{\xi}_n^2$; hereafter we use $\gamma = 3$ throughout the article. Thus, if the population parameter were known, the theoretically optimal sample size would be obtainable, yielding a fixed-n approach for sample size planning. However, in application, intervals are either wider or narrower due to sampling variability of estimates from the study instead of using the known population value upon which n_ω is based. We solve this limitation in the next section with sequential AIPE.

Accuracy in Parameter Estimation Via a Sequential Optimization Procedure

In sequential methodologies, the sample size is not fixed in advance as it is in fixed sample-size procedures. Here we propose a sequential procedure to construct a $(1 - \alpha)100\%$ confidence interval for the Pearson's product-moment correlation coefficient ρ within a distribution-free environment. For details about the general theory of sequential estimation procedures, we refer interested readers to Chattopadhyay and Kelley (2017, 2016), Chattopadhyay and Mukhopadhyay (2013), De and Chattopadhyay (2017), Ghosh and Sen (1991), and Sen (1981). Recall that the optimal sample size n_ω is unknown due to ξ^2 being unknown. We use the consistent estimator of ξ^2, namely $\hat{\xi}_n^2$, which is based on n observations of both X and Y. We now develop an algorithm to find an estimate of the theoretically optimal sample size via the purely sequential estimation procedure.

Stage I

Observations are collected on (paired) variables X and Y for a randomly selected sample of size m, the pilot sample size. We recommend using the pilot sample size, m, following Mukhopadhyay (1980), as

$$m = \max\left\{4, \left\lceil \frac{2z_{\alpha/2}}{\omega} \right\rceil\right\}. \quad (9)$$

Based on this pilot sample of size m, we estimate ξ^2 by computing $\hat{\xi}_m^2$. If $m < \lceil 4z_{\alpha/2}^2/\omega^2(\hat{\xi}_m^2 + m^{-1})\rceil$, then proceed to the next step. Otherwise, if $m \geq \lceil 4z_{\alpha/2}^2/\omega^2(\hat{\xi}_m^2 + m^{-1})\rceil$, stop sampling and set the final sample size equal to m.

Stage II

Obtain an additional $m'(\geq 1)$ observations, where $m'(\geq 1)$ is the number of paired observations that are added to the sample in every stage after the pilot stage. Thus, for adding a single pair to the collected data, $m' = 1$. However, if five additional pairs are taken at each stage, then $m' = 5$. Thus, after collecting the pilot sample and the sampling at the next stage, there are $(m + m')$ observations on both X and Y. After updating the estimate of ξ^2 by computing $\hat{\xi}_{m+m'}^2$, a check is performed to determine whether $m + m' \geq \lceil 4z_{\alpha/2}^2/\omega^2(\hat{\xi}_{m+m'}^2 + (m + m')^{-1})\rceil$. If $m + m' < \lceil 4z_{\alpha/2}^2/\omega^2(\hat{\xi}_{m+m'}^2 + (m + m')^{-1})\rceil$ then go to the next step. Otherwise, if $m + m' \geq \lceil 4z_{\alpha/2}^2/\omega^2(\hat{\xi}_{m+m'}^2 + (m + m')^{-1})\rceil$ then stop further sampling and report that the final sample size is $(m + m')$.

This process of collecting one (or more) observation(s) in each stage after the first stage continues until there are N observations such that $N \geq \lceil (4z_{\alpha/2}^2/\omega^2)(\hat{\xi}_N^2 + N^{-1})\rceil$. At this stage, we stop further sampling and report that the final sample size is N.

Based on the algorithm just outlined, a sampling stopping rule can be defined as follows:

$$N \text{ is the smallest integer } n(\geq m) \text{ such that } n \geq \frac{4z_{\alpha/2}^2}{\omega^2}\left(\hat{\xi}_n^2 + n^{-1}\right), \quad (10)$$

where the term n^{-1} is a correction term ensuring that the sampling process does not stop too early for the optimal sample size because of the use of the approximate expression. The inclusion of the correction term retains the convergence property of $\hat{\xi}_n^2 + n^{-1}$, thus $\hat{\xi}_n^2 + n^{-1}$ converges to ξ^2 for a large sample size. For details of the correction term, refer to Chattopadhyay and De (2016), Chattopadhyay and Kelley (2017, 2016) and Sen and Ghosh (1981).

Following the sequential procedure, the $(1 - \alpha)100\%$ confidence interval for the population Pearson's product-moment correlation coefficient, ρ, is given by

$$\left[r_N - \frac{z_{\alpha/2}\hat{\xi}_N}{\sqrt{N}}, r_N + \frac{z_{\alpha/2}\hat{\xi}_N}{\sqrt{N}}\right]. \quad (11)$$

The width of the $(1 - \alpha)100\%$ confidence interval in Equation 11 will be less than or equal to ω, in accord with our method's specifications. Lemma 1 in Appendix B proves that the estimated sample size from sequential procedure, N, is finite. Also, Theorem 1 in Appendix B proves that the confidence interval achieves the specified coverage probability $1 - \alpha$ asymptotically using N which is the estimate of the smallest possible sample size (n_ω). Here the smallest possible sample size indicates that the sample size required to obtain a sufficiently narrow $(1 - \alpha)100\%$ confidence interval is n_ω. Because n_ω is unknown, using the sequential procedure developed here we can find a consistent estimator, N, of n_ω (proved in Lemma 2). Additionally, Theorem 1 proves that the confidence interval for ρ given in Equation 11 always achieves a sufficiently narrow width (less than or equal ω).

Example

For illustrative purposes, we provide an example inspired by a study on the relation between personality states and work experiences (see Judge, Simon, Hurst, & Kelley, 2014). Personality is generally theorized to have a trait level and a state level, where the trait is a relatively stable characteristic of an individual but where the state varies around the trait level. Of interest in Judge, Simon, Hurst, and Kelley (2014) is the connection of workplace experiences and personality states. Judge et al. (2014) used a 12-item measure of citizenship behavior (both interpersonal and organizational) at work from Lee and Allen (2002) on a 1 (*strongly disagree today*) to 5 (*strongly agree today*) scale and a 13-item measure of neuroticism from Goldberg (1992) on a 1 (*strongly disagree today*) to 5 (*strongly agree today*) scale, among others. Because the day-to-day relationship between workplace experiences and personality was of interest, the questions were framed in terms of *today* instead of *in general*. The mean of the items was taken for a measure of the construct for each day.

In an effort to better understand, using a context similar to that of Judge et al. (2014), we seek to study the strength of the (linear) relationship between citizenship behaviors at work in the morning (before any meetings) and neuroticism at work in the evening (before leaving for the day). Our goal is to obtain an approximate 95% confidence interval for the population correlation coefficient such that the observed correlation is an accurate measure of the population value, where the desired confidence interval width is no larger than 0.10 units. Thus, the value of α (Type I error rate) is 0.05 and the value of ω (desired confidence interval width) is 0.10.

Using the values $\omega(= 0.10)$ and $\alpha(= 0.05)$, we obtain a pilot sample size m using the pilot sample size formula given in Equation 9. Thus, $m = \max\{4, \lceil 2 \times 1.96/0.10\rceil\} = 40$. Using the MBESS R package, the `sq.aipe.cc()` function can be used to obtain the pilot sample as follows:

```
require(MBESS)
sq.aipe.cc(alpha = .05, omega = 0.1, pilot =
   TRUE)
```

where we set R code in typewriter font in order to distinguish the code from regular text. The pilot sample size, $m = 40$, implies that observations on the variables of interest (work stress measure as X

and neuroticism as *Y*) should be taken from 40 randomly selected individuals.

Here we consider hypothetical data that might be collected from the above noted neuroticism and citizenship scales. We show the R code to enter the data, with ". . ." representing data that would be in the full code but is not included in this article:

```
Neuroticism <- c(1.54, 1.00, . . . , 3.08,
  1.00)
Citizenship <- c(2.75, 3.08, . . . , 3.83,
  3.50)
```

Given `Neuroticism` and `Citizenship` objects as defined, we can now use the `sq.aipe.cc()` function, which provides the outputs: (a) current sample size, (b) estimated correlation coefficient, and (c) whether the stopping rule is met or not. If the stopping rule is met, the function additionally provides (d) the confidence interval. The `sq.aipe.cc()` function is used as:

```
sq.aipe.cc(alpha = .05, omega = 0.1, var.1 =
   Neuroticism, var.2 = Citizenship)
```

and the output that we get is as follows:

```
[1] "The stopping rule has not yet been met;
    sample size is not large enough"
$ Current.n
[1] 40
Current.cc
[1] −0.1081764
$'Is.Satisfied?'
[1] FALSE
```

As can be seen, at this point, the output returns `FALSE` with regard to the question "Is the stopping rule met?," indicating that the stopping rule defined in Equation 10 was not satisfied. For now, suppose that $m' = 10$, which means that 10 observations will be added, and then check whether the stopping rule is satisfied.[2] Thus, an updated dataset is obtained by augmenting the first dataset with the additional 10 observations:

```
Neuroticism <- c(1.54, 1.00, . . . , 3.08,
  1.00, . . . , 2.00, 1.15)
Citizenship <- c(2.75, 3.08, . . . , 3.83,
  3.50, . . . , 4.00, 2.67)
```

Using the new data, we apply the function `sq.aipe.cc()` again:

```
sq.aipe.cc(alpha = .05, omega = 0.1, var.1 =
   Neuroticism, var.2 = Citizenship)
```

and the output that we get is as follows:

```
[1] "The stopping rule has not yet been met;
    sample size is not large enough"
$ Current.n
[1] 50
$ Current.cc
[1] −0.1518521
$'Is.Satisfied?'
[1] FALSE
```

This process continues until the stopping rule is satisfied. A sample size of 1,490 leads to a correlation of −0.117 but the stopping rule is not yet satisfied. However, with a sample size of 1,500 the stopping rule is satisfied, and the code is such that when the `sq.aipe.cc()` is used on the data of size 1,500,

```
sq.aipe.cc(alpha = .05, omega = 0.1, var.1 =
   Neuroticism, var.2 = Citizenship)
```

The output that we get is as follows:

```
[1] "The stopping rule has been met"
$ Current.n
[1] 1500
$ Current.cc
[1] −0.1170824
$'Is.Satisfied?'
[1] TRUE
$'Confidence Interval'
[1] "-0.16694, −0.06722"
```

Notice from above that the confidence interval width is 0.09972, which satisfies the originally specified goal (the smallest sample size in which the width is no larger than 0.10 units). Importantly, we did not make distribution assumptions in the construction of the confidence interval. Arguably even more importantly, in order to satisfy our accurate estimate, as specified in terms of confidence interval width, we did not have to prespecify any supposed values of population parameters.

Characteristics of the Final Sample Size for Pearson's Product Moment Correlation: A Simulation Study

Recall that our procedure is asymptotically correct but its effectiveness in smaller sample size situations is not fully known, which is due to the fact that the methods of confidence interval construction are themselves asymptotically correct. Correspondingly, we now demonstrate the properties of our method using a Monte Carlo simulation for constructing $(1 - \alpha)100\%$ confidence intervals for population correlation coefficients from a variety of different bivariate distributions. To implement the sequential AIPE procedure we developed, we specify an example maximum confidence interval width of $\omega = 0.1$ and a confidence coefficient of 90%. We compute the pilot sample size by using the formula given in the algorithm $m = 33$ (= max $\{4, \lceil 2z_{0.1/2}/0.1 \rceil\}$). The estimate of the asymptotic variance of the Pearson's product-moment correlation coefficient is calculated using the pilot sample, and we check if the stopping rule in Equation 10 is satisfied. If the stopping rule is satisfied, the sampling stops. Otherwise, an additional sample of size $m' = 1$ is generated from the specified bivariate distribution and the updated asymptotic variance recalculated. This continues until the stopping rule in Equation 10 is satisfied. The simulation results are based on two different distributions: bivariate normal and bivariate gamma distributions. For bivariate normal and the bivariate gamma distribution from Theorem 2 of Nadarajah and Gupta (2006), the simulation study was done for population Pearson's product-moment correlation coefficients $\rho = \{0.1, 0.3, 0.5\}$, $\omega = \{0.1, 0.2\}$ and $\alpha = \{0.1, 0.05\}$. The values of ρ were chosen to reflect small (0.10), medium (0.30), and large (0.50) effect sizes for correlations (Cohen, 1988, section 3.2). We note that these correlation sizes (along with 0.2 and 0.4, which were not

[2] We use $m' = 10$. However, depending on the situation, m' could be 1 or any larger positive integer. In some situations, for example, it may be easy to collect groups of 10 or 20 or 30 observations at a time (e.g., in a behavioral laboratory) and thus for ease m' might be thought best to be large. However, in other situations, such as when collecting data from Amazon Mechanical Turk, where the data arrives one-by-one, using $m' = 1$ might be a reasonable choice when the process is largely automated.

thought to be needed) were used in Kelley and Maxwell (2003) for a more general regression model. In all simulation conditions, 5,000 replications were used.

Tables 1 and 2 show the mean final sample size $\bar{N}$ (estimates E[N]), coverage probability p, and average confidence interval width $\bar{w}_N$ (estimates E[w_n]). The values $se(\bar{N})$, $se(p)$, and $se(\bar{w}_N)$ represent the standard errors of $\bar{N}$, p, and $\bar{w}_N$ respectively. None of the confidence interval widths, w_N, obtained from the final sample sizes, N, exceeded the maximum specified width, ω. Tables 1 and 2 show that, in most cases (except for smaller sample sizes), the ratio of the mean final sample size to the theoretical sample sizes is satisfactory, if not highly so. However, in some situations, the ratio of the mean final sample size to the theoretical sample sizes is not on target (e.g., <85% empirical coverage in the situation of a 90% confidence interval). These situations, however, occur only when the empirical confidence interval coverage differs markedly from the nominal coverage. We will discuss this limitation below, which is due to the under estimation of ξ^2 by its estimator $\hat{\xi}_n^2$ and not the sequential AIPE procedure itself.

Tables 1 and 2 show that, in most situations, our sequential procedure works well. However, there are some situations where (a) the ratio of the mean final sample size to the theoretical sample size (i.e., $\bar{N}/n_\omega$) is considerably less than 1.0, such as in Table 1 in the last row. There, the ratio of the mean final sample size to the theoretical sample size is 0.82. In particular, the mean final sample size was 164 but the theoretical sample size was 200. However, also note that the confidence interval coverage, nominally set to 90%, was shown to be only 76.86%. Consideration of this issue led to a separate simulation study to investigate the source of the problem, which is discussed in Appendix D. A summary of the discussion in Appendix D is that the undercoverage issue is not due to the sequential AIPE procedure, but rather the confidence interval method that we used with the sequential AIPE procedure. As the sequential AIPE procedure itself is not wedded to the confidence interval approaches used, other methods can be developed and the sequential AIPE procedure applied.

Alternative Confidence Intervals for Pearson's Product Moment Correlation Coefficient

Our sequential procedure developed in the article so far can be extended to other forms of confidence intervals for the population Pearson's product-moment correlation coefficient, such as those proposed by Corty and Corty (2011) and Moinester and Gottfried (2014). We discuss how our methods apply to the methods recommended by these authors. We are agnostic to which method should be used, but rather want to show how our methods work for both situations.

Confidence Interval by Corty and Corty (2011)

Corty and Corty (2011) used Fisher's z-transform and thereby proposed a way to estimate the sample size for a given choice of sample correlation coefficient, confidence level, and ω.

Moinester and Gottfried (2014) noted that the optimal sample size required to achieve a 95% confidence interval for the Pearson's product-moment correlation coefficient, proposed by Corty and Corty (2011), with width no larger than ω, is

$$n_{CC} = \frac{15.37}{(0.5 \times \ln(B))^2} + 3, \tag{12}$$

where

$$B = \frac{(1 + |\rho| + \omega/2)(1 - |\rho| + \omega/2)}{(1 + |\rho| - \omega/2)(1 - |\rho| - \omega/2)}, \tag{13}$$

and $|\rho|$ is the absolute value of the population correlation coefficient. The supposed value of the population Pearson's product-moment correlation coefficient, whose confidence interval we would like to construct, can differ markedly from the true population value. As discussed, our sequential procedure does not require inserting supposed population values a priori. Our sequential stopping rule which helps find the estimate of the optimal sample size is as follows:

N_{CC} is the smallest integer $n(\geq m_{CC})$ such that

Table 1
Summary of Final Sample Size for 90% Confidence Interval for ρ

ω	Distribution	ρ	$\bar{N}$	$se(\bar{N})$	n_ω	$\bar{N}/n_\omega$	p	s_p	$\bar{w}_N$	$se(\bar{w}_N)$
.1	$N_2(0, 0, 1, 1, .1)$	.1	1056.0	.9581	1,061	.9957	.8944	.0043	.0999	9.33×10^{-7}
	$N_2(0, 0, 1, 1, .3)$	.3	891.2	.9900	897	.9935	.8902	.0044	.0999	1.21×10^{-6}
	$N_2(0, 0, 1, 1, .5)$	.5	601.0	.9920	609	.9869	.8868	.0045	.0997	2.04×10^{-6}
	$Ga_2(5, 5, 50, 10)$	.1	1124.0	1.5480	1,138	.9876	.8984	.0043	.0999	1.66×10^{-6}
	$Ga_2(5, 5, 16.67, 10)$	.3	1049.0	1.6510	1,066	.9840	.8982	.0043	.0999	1.97×10^{-6}
	$Ga_2(5, 5, 10, 10)$	.5	774.4	1.6730	797	.9717	.8862	.0045	.0995	7.236×10^{-5}
.2	$N_2(0, 0, 1, 1, .1)$	.1	260.3	.5085	266	.9784	.8762	.0047	.1985	1.48×10^{-4}
	$N_2(0, 0, 1, 1, .3)$	.3	216.4	.6056	225	.9617	.8666	.0048	.1963	2.86×10^{-4}
	$N_2(0, 0, 1, 1, .5)$	.5	138.0	.6628	153	.9022	.8124	.0055	.1873	5.60×10^{-4}
	$Ga_2(5, 5, 50, 10)$	.1	272.0	.7694	285	.9545	.8784	.0046	.1978	2.071×10^{-4}
	$Ga_2(5, 5, 16.67, 10)$	.3	245.9	.9045	267	.9209	.8546	.0050	.1947	3.69×10^{-4}
	$Ga_2(5, 5, 10, 10)$	.5	164.0	1.0100	200	.8198	.7686	.0060	.1806	7.25×10^{-4}

Note. ρ is the population correlation coefficient; $\bar{N}$ is the mean final sample size; p is the estimated coverage probability; ω is the upper bound of the length of the confidence interval for ρ; $se(\bar{N})$ is the standard deviation of the mean final sample size (i.e., standard error of the final sample size); n_ω is the theoretical sample size if the procedure is used with the population parameters; $se(p)$ is the standard error of p; $\bar{w}_N$ is the average length of confidence intervals for ρ based on N observations; $se(\bar{w}_N)$ is the standard error of $\bar{w}$; tabled values are based on 5,000 replications of a Monte Carlo simulation study from distributions: bivariate normal (N_2) distribution with parameters μ_1, μ_2, σ_1^2, σ_2^2, and ρ, respectively, and bivariate gamma (Ga_2) with parameters a_1, a_2, c, and μ, respectively, based on Theorem 2 of Nadarajah and Gupta (2006).

Table 2
Summary of Final Sample Size for 95% Confidence Interval for ρ

ω	Distribution	ρ	$\bar{N}$	$se(\bar{N})$	n_ω	$\bar{N}/n_\omega$	p	$se(p)$	$\bar{w}_N$	$se(\bar{w}_N)$
.1	$N_2(0, 0, 1, 1, .1)$	.1	1502.0	1.1200	1,507	.9969	.9442	.0032	.0999	6.44×10^{-7}
	$N_2(0, 0, 1, 1, .3)$	.3	1267.2	1.1890	1,273	.9956	.9460	.0032	.0999	8.13×10^{-7}
	$N_2(0, 0, 1, 1, .5)$	.5	857.0	1.1920	865	.9908	.9464	.0032	.0998	7.91×10^{-6}
	$Ga_2(5, 5, 50, 10)$	.1	1600.0	1.8780	1,615	.9910	.9498	.0031	.0999	1.23×10^{-7}
	$Ga_2(5, 5, 16.67, 10)$	.3	1497.0	1.9960	1,513	.9840	.9490	.0031	.0999	1.24×10^{-6}
	$Ga_2(5, 5, 10, 10)$	.5	1109.0	1.9540	1,132	.9794	.9430	.0033	.0998	3.90×10^{-5}
.2	$N_2(0, 0, 1, 1, .1)$	.1	372.2	.5746	377	.9872	.9396	.0034	.1992	8.16×10^{-5}
	$N_2(0, 0, 1, 1, .3)$	.3	312.5	.6112	319	.9796	.9332	.0035	.1989	9.71×10^{-5}
	$N_2(0, 0, 1, 1, .5)$	.5	204.0	.7435	217	.9400	.9012	.0042	.1938	3.83×10^{-4}
	$Ga_2(5, 5, 50, 10)$	.1	391.1	.8966	404	.9681	.9412	.0033	.1989	1.15×10^{-4}
	$Ga_2(5, 5, 16.67, 10)$	.3	360.1	1.0160	379	.9501	.9252	.0037	.1977	2.30×10^{-4}
	$Ga_2(5, 5, 10, 10)$	.5	251.6	1.164	283	.8889	.8766	.0047	.1901	5.3×10^{-4}

Note. ρ is the population correlation coefficient; $\bar{N}$ is the mean final sample size; p is the estimated coverage probability; ω is the upper bound of the length of the confidence interval for ρ; $se(\bar{N})$ is the standard deviation of the mean final sample size (i.e., standard error of the final sample size); n_ω is the theoretical sample size if the procedure is used with the population parameters; $se(p)$ is the standard error of p; $\bar{w}_N$ is the average length of confidence intervals for ρ based on N observations; $se(\bar{w}_N)$ is the standard error of $\bar{w}$; tabled values are based on 5,000 replications of a Monte Carlo simulation study from distributions: bivariate normal (N_2) distribution with parameters μ_1, μ_2, σ_1^2, σ_2^2, and ρ, respectively, and bivariate gamma (Ga_2) with parameters a_1, a_2, c, and μ, respectively, based on Theorem 2 of Nadarajah and Gupta (2006).

$$n \geq 61.48\left(\left(\ln\left(\frac{(1+|r_n|+\omega/2)(1-|r_n|+\omega/2)}{(1+|r_n|-\omega/2)(1-|r_n|-\omega/2)}\right)\right)^{-2}+\frac{1}{n}\right)+3. \tag{14}$$

After the stopping rule is satisfied, the 95% confidence interval for the population Pearson's product-moment correlation coefficient, ρ, can be constructed by applying the confidence interval formula as in Corty and Corty (2011). We suggest the pilot sample size, m_{CC}, as

$$m_{CC} = \max\left\{4, \left[\frac{1}{2}\left[3+\frac{16z_{\alpha/2}^2}{(\ln b)^2}+\sqrt{\left(3+\frac{16z_{\alpha/2}^2}{(\ln b)^2}\right)^2+(8z_{\alpha/2})^2}\right]\right]\right\}, \tag{15}$$

where

$$b = \frac{\left(2+\frac{\omega}{2}\right)\left(1+\frac{\omega}{2}\right)}{\left(1-\frac{\omega}{2}\right)\left(\frac{1-2\omega}{4}\right)} \tag{16}$$

for $r < 1 - \frac{\omega}{2}$ and $\omega < 0.5$.

The derivation of the pilot sample size given in Equation 15 is shown in Appendix C. The optimal sample size, n_{CC}, can be estimated by adopting the sequential stopping rule defined in Equation 14. Note that, in practice, n_{CC} is usually unknown because ρ will usually be unknown. Thus, when one uses supposed values of parameters a final sample size is known, but the value is almost certainly not the correct value given uncertainty in ρ a priori.

Confidence Interval for ρ With Moinester and Gottfried's (2014) Method

In Method 4 of Moinester and Gottfried (2014), the 95% confidence interval for the population correlation coefficient, when observations are assumed to be from a bivariate-normal distribution, is

$$\left[r_n - 1.96\sqrt{\frac{1-r_n^2}{n-1}}, r_n + 1.96\sqrt{\frac{1-r_n^2}{n-1}}\right]. \tag{17}$$

The optimal sample size required to achieve a 95% confidence interval for correlation coefficient (ρ) with width no larger than ω is

$$n_{MG} = \frac{3.84(1-\rho^2)^2}{(\omega/2)^2}+1. \tag{18}$$

Our sequential stopping rule, which does not take into account the supposed value of the population correlation coefficient, is as follows:

N_{MG} is the smallest integer $n(\geq m_{MG})$ such that

$$n \geq \frac{15.36}{\omega^2}((1-r_n^2)^2+1/n)+1. \tag{19}$$

Following the sample of size N_{MG} collected using the sequential stopping rule of Equation 19, the 95% confidence interval for ρ is

$$\left[r_{N_{MG}} - 1.96\sqrt{\frac{1-r_{N_{MG}}^2}{N_{MG}-1}}, r_{N_{MG}} + 1.96\sqrt{\frac{1-r_{N_{MG}}^2}{N_{MG}-1}}\right]. \tag{20}$$

We suggest the pilot sample size of

$$m_{MG} = \max\left\{4, \left[\frac{1+\sqrt{1+(4z_{\alpha/2}/\omega)^2}}{2}\right]\right\}. \tag{21}$$

The derivation of the pilot sample size given in Equation 21 is shown in Appendix C. The optimal sample size, m_{MG}, can be estimated by following the sequential stopping rule defined in Equation 19.

Simulation Study

We now compare the characteristics of the stopping rule defined in Equation 10 with the stopping rules defined in Equations 14 and 19 using a Monte Carlo simulation study for constructing $(1-\alpha)100\%$ confidence intervals for population correlation coefficients from bivariate distributions. For bivari-

Table 3
Summary of Final Sample Size for 90% Confidence Interval for ρ *Using Corty and Corty (2011)*

ω	Distribution	ρ	$\bar{N}_{CC}$	$se(\bar{N}_{CC})$	n_{CC}	$\bar{N}_{CC}/n_{CC}$	p_{CC}	$se(p_{CC})$	$\bar{w}_{N_{CC}}$	$se(\bar{w}_{N_{CC}})$
.1	$N_2(0, 0, 1, 1, .1)$	.1	1060.0	.1942	1,062	.9979	.8980	.0043	.1000	2.75×10^{-7}
	$N_2(0, 0, 1, 1, .3)$	.3	893.6	.5208	897	.9962	.8986	.0043	.1000	7.71×10^{-7}
	$N_2(0, 0, 1, 1, .5)$	.5	604.4	.7567	609	.9924	.8868	.0045	.1001	1.32×10^{-5}
	$Ga_2(5, 5, 50, 10)$	.1	1060.0	.1966	1,062	.9982	.8906	.0044	.1000	3.03×10^{-7}
	$Ga_2(5, 5, 16.67, 10)$	.3	894.5	.5549	897	.9972	.8728	.0047	.1000	1.19×10^{-6}
	$Ga_2(5, 5, 10, 10)$	.5	604.5	.8638	609	.9925	.8464	.0051	.1001	1.52×10^{-5}
.2	$N_2(0, 0, 1, 1, .1)$	.1	264.6	.1173	267	.9909	.8892	.0044	.1999	1.16×10^{-5}
	$N_2(0, 0, 1, 1, .3)$	.3	222.1	.2994	225	.9872	.8916	.0044	.2002	3.7×10^{-5}
	$N_2(0, 0, 1, 1, .5)$	.5	146.1	.4565	153	.9551	.8692	.0048	.2021	1.14×10^{-4}
	$Ga_2(5, 5, 50, 10)$	.1	264.1	.1665	267	.9892	.8842	.0045	.1999	3.10×10^{-5}
	$Ga_2(5, 5, 16.67, 10)$	.3	221.8	.3441	225	.9859	.8606	.0049	.2002	5.22×10^{-5}
	$Ga_2(5, 5, 10, 10)$	.5	143.1	.5639	153	.9351	.7942	.0057	.2027	1.54×10^{-4}

Note. ρ is the population correlation coefficient; $\bar{N}_{CC}$ is the mean final sample size; p_{CC} is the estimated coverage probability; ω is the upper bound of the length of the confidence interval for ρ; $se(\bar{N}_{CC})$ is the standard deviation of the mean final sample size (i.e., standard error of the final sample size); n_{CC} is the theoretical sample size if the procedure is used with the population parameters; $se(p_{CC})$ is the standard error of p_{CC}; $\bar{w}_{N_{CC}}$ is the average length of confidence intervals for ρ based on N_{CC} observations; $se(\bar{w}_{N_{CC}})$ is the standard error of $\bar{w}_{N_{CC}}$; tabled values are based on 5,000 replications of a Monte Carlo simulation study from distributions: bivariate normal (N_2) distribution with parameters μ_1, μ_2, σ_1^2, σ_2^2, and ρ, respectively, and bivariate gamma (Ga_2) with parameters a_1, a_2, c, and μ, respectively, based on Theorem 2 of Nadarajah and Gupta (2006).

ate normal and the bivariate gamma distribution from Theorem 2 of Nadarajah and Gupta (2006), the simulation study was done for Pearson's product-moment correlation coefficients ρ = {0.1, 0.3, 0.5} and ω = {0.1, 0.2}. In all cases, 5,000 replications were used. Tables 3, 4, 5, and 6 shows the estimates of mean final sample size, coverage probability, and average confidence interval width and also the corresponding standard errors.

Comparing the characteristics of the stopping rule defined in Equation 10 with the stopping rules defined in Equations 14 and 19, we observe that the behavior of the coverage probability as well as ratio of average sample size estimate and the optimal sample size are similar in all three procedures.

Sequential AIPE for Kendall's Tau and Spearman's Rho

We now discuss the sequential approach related to the accuracy in parameter estimation problem for estimating Kendall's rank correlation coefficient, popularly known as Kendall's tau and denoted here by τ, and Spearman's rank correlation coefficient, popularly known as Spearman's rho, and denoted here by ρ_s.

AIPE for Kendall's τ

Kendall's τ is a statistic which can be used to measure the ordinal association between two variables. Suppose (X, Y) denote a pair of random observations with a joint distribution function F.

Table 4
Summary of Final Sample Size for 95% Confidence Interval for ρ *Using Corty and Corty (2011)*

ω	Distribution	ρ	$\bar{N}_{CC}$	$se(\bar{N}_{CC})$	n_{CC}	$\bar{N}_{CC}/n_{CC}$	p_{CC}	$se(p_{CC})$	$\bar{w}_{N_{CC}}$	$se(\bar{w}_{N_{CC}})$
.1	$N_2(0, 0, 1, 1, .1)$	.1	1,504.0	.2268	1,507	.9983	.9474	.0032	.1000	1.915×10^{-7}
	$N_2(0, 0, 1, 1, .3)$	.3	1,269.0	.6185	1,273	.9971	.9462	.0032	.1000	5.64×10^{-7}
	$N_2(0, 0, 1, 1, .5)$	.5	857.6	.8464	863	.9938	.9452	.0032	.1001	1.43×10^{-6}
	$Ga_2(5, 5, 50, 10)$	.1	1,505.0	.2318	1,507	.9985	.9400	.0034	.1000	2.10×10^{-7}
	$Ga_2(5, 5, 16.67, 10)$	.3	1,270.0	.6581	1,273	.9975	.9354	.0035	.1000	8.25×10^{-7}
	$Ga_2(5, 5, 10, 10)$	.5	859.1	.9800	863	.9955	.9090	.0041	.1001	9.37×10^{-6}
.2	$N_2(0, 0, 1, 1, .1)$	.1	375.2	.1206	377	.9952	.9478	.0031	.1999	1.69×10^{-6}
	$N_2(0, 0, 1, 1, .3)$	.3	315.6	.3343	318	.9925	.9444	.0032	.2002	2.81×10^{-5}
	$N_2(0, 0, 1, 1, .5)$	.5	210.1	.4924	215	.9774	.9400	.0034	.2017	7.54×10^{-5}
	$Ga_2(5, 5, 50, 10)$	.1	375.1	.1250	377	.9950	.9418	.0033	.1999	1.93×10^{-6}
	$Ga_2(5, 5, 16.67, 10)$	.3	315.7	.3694	318	.9926	.9248	.0037	.2002	2.60×10^{-5}
	$Ga_2(5, 5, 10, 10)$	.5	208.2	.5970	215	.9686	.8866	.0045	.2019	9.94×10^{-5}

Note. ρ is the population correlation coefficient; $\bar{N}_{CC}$ is the mean final sample size; p_{CC} is the estimated coverage probability; ω is the upper bound of the length of the confidence interval for ρ; $se(\bar{N}_{CC})$ is the standard deviation of the mean final sample size (i.e., standard error of the final sample size); n_{CC} is the theoretical sample size if the procedure is used with the population parameters; $se(p_{CC})$ is the standard error of p_{CC}; $\bar{w}_{N_{CC}}$ average length of confidence intervals for ρ based on N_{CC} observations; $se(\bar{w}_{N_{CC}})$ is the standard error of $\bar{w}_{N_{CC}}$; tabled values are based on 5,000 replications of a Monte Carlo simulation study from distributions: bivariate normal (N_2) distribution with parameters μ_1, μ_2, σ_1^2, σ_2^2, and ρ, respectively, and bivariate gamma (Ga_2) with parameters a_1, a_2, c, and μ, respectively, based on Theorem 2 of Nadarajah and Gupta (2006).

Table 5
Summary of Final Sample Size for 90% Confidence Interval for ρ *Using Moinester and Gottfried (2014)*

ω	Distribution	ρ	$\bar{N}_{MG}$	$se(\bar{N}_{MG})$	n_{MG}	$\bar{N}_{MG}/n_{MG}$	p_{MG}	$se(p_{MG})$	$\bar{w}_{N_{MG}}$	$se(\bar{w}_{N_{MG}})$
.1	$N_2(0, 0, 1, 1, .1)$	.1	1,060.0	.1936	1,062	.9978	.8944	.0043	.1000	2.74×10^{-7}
	$N_2(0, 0, 1, 1, .3)$	.3	893.9	.5182	898	.9955	.8990	.0043	.0999	7.62×10^{-7}
	$N_2(0, 0, 1, 1, .5)$	.5	606.1	.7103	610	.9935	.8932	.0044	.0999	1.90×10^{-6}
.2	$N_2(0, 0, 1, 1, .1)$	.1	264.5	.1032	267	.9907	.8868	.0045	.1998	2.37×10^{-6}
	$N_2(0, 0, 1, 1, .3)$	.3	222.7	.2779	226	.9853	.8848	.0045	.1995	8.62×10^{-6}
	$N_2(0, 0, 1, 1, .5)$	.5	149.1	.4043	154	.9681	.8642	.0048	.1989	2.21×10^{-5}

Note. ρ is the population correlation coefficient; $\bar{N}_{MG}$ is the mean final sample size; p_{MG} is the estimated coverage probability; ω is the upper bound of the length of the confidence interval for ρ; $se(\bar{N}_{MG})$ is the standard deviation of the mean final sample size (i.e., standard error of the final sample size); n_{MG} is the theoretical sample size if the procedure is used with the population parameters; $se(p_{MG})$ is the standard error of p_{MG}; $\bar{w}_{N_{MG}}$ average length of confidence intervals for ρ based on N_{MG} observations; $se(\bar{w}_{N_{MG}})$ is the standard error of $\bar{w}_{N_{MG}}$; tabled values are based on 5,000 replications of a Monte Carlo simulation study from bivariate normal distribution (N_2) with parameters: means, variances, and correlation.

If (X_1, Y_1) and (X_2, Y_2) are random bivariate observations from F, then Kendall's tau which measures the association between variables X and Y can be defined as

$$\tau = \mathrm{E}[\mathrm{sgn}(X_1 - X_2)\mathrm{sgn}(Y_1 - Y_2)] \tag{22}$$

where

$$\mathrm{sgn}(x) = \begin{cases} -1, & \text{when } x < 0, \\ 0 & \text{when } x = 0, \\ +1 & \text{when } x < 0. \end{cases}$$

An estimator of Kendall's τ is given by

$$r_{\tau,n} = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} \mathrm{sgn}(X_i - X_j)\mathrm{sgn}(Y_i - Y_j), \tag{23}$$

which is a U-statistic (see Lee, 1990). Hoeffding (1948) as well as Daniels and Kendall (1947) have shown that the asymptotic distribution of τ, defined in Equation 22, is given by

$$\sqrt{n}(r_{\tau,n} - \tau) \xrightarrow{D} N(0, \xi_\tau^2), \tag{24}$$

where the expression of the asymptotic variance, ξ_τ^2, is given by

$$\begin{aligned} \xi_\tau^2 &= 4\mathrm{Var}\{\mathrm{E}[\mathrm{sgn}(X_1 - X_2)\mathrm{sgn}(Y_1 - Y_2)|X_1, Y_1]\} \\ &= 4\mathrm{Var}\{1 - 2F_1(X_1) - 2F_2(Y_1) + 4F(X_1, Y_1)\}, \end{aligned} \tag{25}$$

provided F_1 and F_2 are the marginal distributions of X and Y, respectively. Proceeding along the same lines as in Equations 4–7, we can find that the sample size required to achieve the sufficient accuracy with prespecified upper bound (ω) on the width of the confidence interval for τ will be

$$n \ge \left\lceil \frac{4z_{\alpha/2}^2 \xi_\tau^2}{\omega^2} \right\rceil \equiv n_{KT}, \tag{26}$$

where ξ_τ^2 is defined as in Equation 25. In reality, ξ_τ^2 is unknown, so we use a consistent estimator, which is given by

$$\hat{\xi}_{n,KT}^2 = \frac{16}{n-1} \sum_{i=1}^{n} (W_i - \bar{w})^2, \tag{27}$$

where

$$W_i = \frac{2}{n} \sum_{k=1}^{n} 1\{R_{x,k} \le R_{x,i}, R_{y,k} \le R_{y,i}\} - \frac{R_{x,i}}{n+1} - \frac{R_{y,i}}{n+1}, \tag{28}$$

$$\bar{W} = \frac{1}{n} \sum_{i=1}^{n} W_i \tag{29}$$

with $1\{A\}$ denoting the indicator function of Set A, and $R_{x,i}$ and $R_{y,i}$ are respectively ranks of X_i among all X's and Y_i among all Y's (e.g., Genest & Favre, 2007; Kojadinovic & Yan, 2010). Because ξ_τ^2 is unknown in reality, in order to compute the required sample size, n_{KT}, we use the sequential procedure outlined previously, but here applied to Kendall's tau. Our sequential stopping rule which helps find the estimate of the optimal sample size is as follows:

Table 6
Summary of Final Sample Size for 95% Confidence Interval for ρ *Using Moinester and Gottfried (2014)*

ω	Distribution	ρ	$\bar{N}_{MG}$	$se(\bar{N}_{MG})$	n_{MG}	$\bar{N}_{MG}/n_{MG}$	p_{MG}	$se(p_{MG})$	$\bar{w}_{N_{MG}}$	$se(\bar{w}_{N_{MG}})$
.1	$N_2(0, 0, 1, 1, .1)$	.1	1505.0	.2233	1,508	.9982	.9498	.0032	.1000	1.94×10^{-7}
	$N_2(0, 0, 1, 1, .3)$	.3	1270.0	.6142	1,274	.9970	.9452	.0032	.1000	5.13×10^{-7}
	$N_2(0, 0, 1, 1, .5)$	.5	860.5	.8382	866	.9937	.9454	.0032	.0999	1.33×10^{-6}
.2	$N_2(0, 0, 1, 1, .1)$	.1	375.8	.1204	378	.9942	.9440	.0033	.1998	1.65×10^{-6}
	$N_2(0, 0, 1, 1, .3)$	.3	317.1	.3343	320	.9908	.9452	.0032	.1997	4.26×10^{-6}
	$N_2(0, 0, 1, 1, .5)$	.5	213.9	.4407	218	.9812	.9376	.0034	.1992	2.66×10^{-5}

Note. ρ is the population correlation coefficient; $\bar{N}_{MG}$ is the mean final sample size; p_{MG} is the estimated coverage probability; ω is the upper bound of the length of the confidence interval for ρ; $se(\bar{N}_{MG})$ is the standard deviation of the mean final sample size (i.e., standard error of the final sample size); n_{MG} is the theoretical sample size if the procedure is used with the population parameters; $se(p_{MG})$ is the standard error of p_{MG}; $\bar{w}_{N_{MG}}$ average length of confidence intervals for ρ based on N_{MG} observations; $se(\bar{w}_{N_{MG}})$ is the standard error of $\bar{w}_{N_{MG}}$; tabled values are based on 5,000 replications of a Monte Carlo simulation study from bivariate normal distribution (N_2) with parameters μ_1, μ_2, σ_1^2, σ_2^2, and ρ, respectively.

N_{KT} is the smallest integer $n(\geq m_{KT})$ such that $\frac{4z_{\alpha/2}^2}{\omega^2}\left(\hat{\xi}_{n,KT}^2 + n^{-1}\right)$

(30)

where m_{KT} is the pilot sample, the same as that given in Equation 9.

We now find the characteristics of the stopping rule defined in Equation 30 using Monte Carlo simulation for constructing $(1 - \alpha)100\%$ confidence intervals for population correlation coefficients from bivariate distributions — bivariate normal and the bivariate gamma distribution from Theorem 2 of Nadarajah and Gupta (2006). The simulation study was done for correlation coefficient τ corresponding to $\rho = \{0.1, 0.3, 0.5\}$ and $\omega = \{0.1, 0.2\}$. In all cases, 5,000 replications were used. Tables 7 and 8 show the estimates of mean final sample size, coverage probability, and average confidence interval width and also the corresponding standard errors for 90% and 95% confidence interval coverage, respectively.

The width of the confidence interval given by the sequential procedure with stopping rule defined in Equation 30 did not exceed the maximum specified width ω. Further, the coverage probability estimates are close to the corresponding confidence level. Also, the ratio of average sample size estimate and the optimal sample size is close to 1.

AIPE for Spearman's ρ

Let (X, Y) be a random bivariate observation with common distribution function F with marginals $F_1(x)$ and $F_2(y)$, respectively, for X and Y. The popular nonparametric correlation measure proposed by Spearman (1904), which is equivalent to the Pearson correlation for the ranks of observations, is defined as

$$\rho_s = \mathrm{Corr}(F_1(X)F_2(Y)) = 12\mathrm{E}\left[F_1(X)F_2(Y)\right] - 3. \quad (31)$$

For more details, we refer to Borkowf (1999), Croux and Dehon (2010), Genest and Favre (2007), and Kojadinovic and Yan (2010). A consistent estimator for Spearman's ρ, ρ_s based on observations $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$, is given by

$$r_{s,n} = \frac{\sum_{i=1}^{n}(R_{x,i} - \overline{R}_x)(R_{y,i} - \overline{R}_y)}{\sqrt{\sum_{i=1}^{n}(R_{x,i} - \overline{R}_x)^2 \sum_{i=1}^{n}(R_{y,i} - \overline{R}_y)^2}} \quad (32)$$

$$= \frac{12}{n(n+1)(n-1)}\sum_{i=1}^{n} R_{x,i}R_{y,i} - 3\frac{n+1}{n-1} \quad (33)$$

$$= 1 - \frac{6\sum_{i=1}^{n}(R_{x,i} - R_{y,i})^2}{n^3 - n}, \quad (34)$$

where $\overline{R}_x$ and $\overline{R}_y$ are the average of the ranks for X and Y, respectively. Using Borkowf (2002) and Hoeffding (1948), the asymptotic distribution of $r_{s,n}$ is

$$\sqrt{n}(r_{s,n} - \rho_s) \xrightarrow{D} N\left(0, \xi_{\rho_s}^2\right), \quad (35)$$

where

$$\xi_{\rho_s}^2 = 144\left(-9\theta_1^2 + \theta_3 + 2\theta_4 + 2\theta_5 + 2\theta_6\right), \quad (36)$$

and

$$\theta_1 = \mathrm{E}\left[F_1(X_1)F_2(Y_1)\right] \quad (37)$$

$$\theta_3 = \mathrm{E}\left[S_1(X_1)^2 S_2(Y_1)^2\right] \quad (38)$$

$$\theta_4 = \mathrm{E}\left[S(X_1, Y_2)S_1(X_2)S_2(Y_1)\right] \quad (39)$$

$$\theta_5 = \mathrm{E}\left[S(\max\{X_1, X_2\})S_2(Y_1)S_2(Y_2)\right] \quad (40)$$

$$\theta_6 = \mathrm{E}\left[S_1(X_1)S_1(X_2)S(\max\{Y_1, Y_2\})\right] \quad (41)$$

$$S_i(x) = 1 - F_i(x), \quad i \in \{1, 2\} \quad (42)$$

$$S(x, y) = 1 - F_1(x) - F_2(y) + F(x, y). \quad (43)$$

Proceeding along the same lines as given in Equations 4–7, we can find that the sample size required to achieve the sufficient

Table 7
Summary of Final Sample Size for 90% Confidence Interval for Kendall's τ Using Asymptotic Distribution

ω	Distribution	ρ	τ	$\bar{N}_{KT}$	$se(\bar{N}_{KT})$	n_{KT}	$\bar{N}_{KT}/n_{KT}$	p_{KT}	$se(p_{KT})$	$\bar{w}_{N_{KT}}$	$se(\bar{w}_{N_{KT}})$
.1	$N_2(0, 0, 1, 1, .1)$	.1	.0638	478.4	.0597	477	1.003	.8946	.0043	.0997	4.815×10^{-7}
	$N_2(0, 0, 1, 1, .3)$	.3	.1940	443.0	.1669	442	1.002	.8888	.0044	.0997	8.224×10^{-7}
	$N_2(0, 0, 1, 1, .5)$	.5	.3333	371.1	.2575	369	1.006	.8876	.0045	.0995	1.562×10^{-6}
	$Ga_2(5, 5, 50, 10)$	.1	.0638	478.4	.0626	477	1.003	.8984	.0043	.0997	4.901×10^{-7}
	$Ga_2(5, 5, 16.67, 10)$	.3	.1940	443.1	.1783	442	1.002	.8952	.0043	.0996	1.022×10^{-6}
	$Ga_2(5, 5, 10, 10)$	.5	.333	370.9	.2881	369	1.005	.8930	.0044	.0995	2.2×10^{-6}
.2	$N_2(0, 0, 1, 1, .1)$	.1	.0638	120.9	.0354	120	1.008	.8824	.0046	.1977	3.834×10^{-6}
	$N_2(0, 0, 1, 1, .3)$	.3	.1904	112.2	.0846	111	1.011	.8840	.0045	.1972	7.805×10^{-6}
	$N_2(0, 0, 1, 1, .5)$	.5	.3333	94.46	.1250	93	1.016	.8862	.0045	.1960	1.615×10^{-5}
	$Ga_2(5, 5, 50, 10)$	.1	.0638	120.9	.0364	120	1.008	.8858	.0045	.1977	4.113×10^{-6}
	$Ga_2(5, 5, 16.67, 10)$	.3	.1940	112.0	.0915	111	1.009	.8866	.0045	.1972	1.037×10^{-5}
	$Ga_2(5, 5, 10, 10)$	.5	.3333	94.16	.1457	93	1.012	.8756	.0047	.1958	2.273×10^{-5}

Note. ρ is the population Pearson's correlation coefficient; τ is the population Kendall's τ (computed using bootstrap method for bivariate Gamma distribution); $\bar{N}_{KT}$ is the mean final sample size; p_{KT} is the estimated coverage probability; ω is the upper bound of the length of the confidence interval for τ; $se(\bar{N}_{KT})$ is the standard deviation of the mean final sample size (i.e., standard error of the final sample size); n_{KT} is the theoretical sample size if the procedure is used with the population parameters; $se(p_{KT})$ is the standard error of p_{KT}; $\bar{w}_{N_{KT}}$ average length of confidence intervals for ρ based on N observations; tabled values are based on 5,000 replications of a Monte Carlo simulation study from distributions: Bivariate Normal (N_2) with parameters μ_1, μ_2, σ_1^2, σ_2^2, and ρ respectively, and bivariate gamma (Ga_2) with parameters a_1, a_2, c, and μ, respectively, based on Theorem 2 of Nadarajah and Gupta (2006).

Table 8
Summary of Final Sample Size for 95% Confidence Interval for Kendall's τ Using Asymptotic Distribution

ω	Distribution	ρ	τ	$\bar{N}_{KT}$	$se(\bar{N}_{KT})$	n_{KT}	$\bar{N}_{KT}/n_{KT}$	p_{KT}	$se(p_{KT})$	$\bar{w}_{N_{KT}}$	$se(\bar{w}_{N_{KT}})$
.1	$N_2(0, 0, 1, 1, .1)$	.1	.0638	678.4	.0698	677	1.002	.9506	.0031	.0998	3.429×10^{-7}
	$N_2(0, 0, 1, 1, .3)$	.3	.1940	628.2	.1984	627	1.002	.9408	.0033	.0998	5.691×10^{-7}
	$N_2(0, 0, 1, 1, .5)$	.5	.3333	526.0	.3058	524	1.004	.9444	.0032	.0996	1.084×10^{-6}
	$Ga_2(5, 5, 50, 10)$	.1	.6380	678.4	.0741	677	1.002	.9490	.0031	.0998	3.41×10^{-7}
	$Ga_2(5, 5, 16.67, 10)$	.3	.1940	628.2	.2084	627	1.002	.9512	.0030	.0998	7.487×10^{-7}
	$Ga_2(5, 5, 10, 10)$	.5	.3333	526.1	.3452	524	1.004	.9394	.0034	.0996	1.539×10^{-6}
.2	$N_2(0, 0, 1, 1, .1)$	.1	.0638	170.9	.0371	170	1.005	.9440	.0033	.1984	3.028×10^{-6}
	$N_2(0, 0, 1, 1, .3)$	.3	.1940	158.2	.1004	157	1.010	.9378	.0034	.1980	5.218×10^{-6}
	$N_2(0, 0, 1, 1, .5)$	.5	.3333	133.2	.1514	131	1.017	.9318	.0036	.1972	1.045×10^{-5}
	$Ga_2(5, 5, 50, 10)$	.1	.638	170.8	.0425	170	1.005	.9428	.0033	.1984	3.335×10^{-6}
	$Ga_2(5, 5, 16.67, 10)$	.3	.194	158.3	.1121	157	1.009	.9364	.0035	.198	6.536×10^{-6}
	$Ga_2(5, 5, 10, 10)$	.5	.333	133.1	.1725	131	1.016	.9332	.0035	.1971	1.47×10^{-5}

Note. ρ is the population Pearson's correlation coefficient; τ is the population Kendall's τ (computed using bootstrap method for bivariate Gamma distribution); $\bar{N}_{KT}$ is the mean final sample size; p_{KT} is the estimated coverage probability; ω is the upper bound of the length of the confidence interval for τ; $se(\bar{N}_{KT})$ is the standard deviation of the mean final sample size (i.e., standard error of the final sample size); n_{KT} is the theoretical sample size if the procedure is used with the population parameters; $se(p_{KT})$ is the standard error of p_{KT}; $\bar{w}_{N_{KT}}$ average length of confidence intervals for ρ based on N observations; tabled values are based on 5,000 replications of a Monte Carlo simulation study from distributions: bivariate normal (N^2) with parameters μ_1, μ_2, σ_1^2, σ_2^2, and ρ, respectively, and bivariate gamma (Ga_2) with parameters a_1, a_2, c, and μ based on Theorem 2 of Nadarajah and Gupta (2006).

accuracy with prespecified upper bound (ω) on the width of the confidence interval for ρ_s will be

$$n \geq \left\lceil \frac{4 z_{\alpha/2}^2 \xi_{\rho_s}^2}{\omega^2} \right\rceil \equiv n_{SR}, \tag{44}$$

where $\xi_{\rho_s}^2$ is defined as in Equation 36. In practice, $\xi_{\rho_s}^2$ is usually unknown and we use a consistent estimator proposed by Genest and Favre (2007), which is given by

$$\hat{\xi}_{n,SR}^2 = 144(-9\mathcal{A}_n + \mathcal{B}_n + 2\mathcal{C}_n + 2\mathcal{D}_n + 2\mathcal{E}_n) \tag{45}$$

where

$$\mathcal{A}_n = \frac{1}{n}\sum_{i=1}^{n} \frac{R_{x,i}}{n+1}\frac{R_{y,i}}{n+1} \tag{46}$$

$$\mathcal{B}_n = \frac{1}{n}\sum_{i=1}^{n} \left(\frac{R_{x,i}}{n+1}\right)^2 \left(\frac{R_{y,i}}{n+1}\right)^2 \tag{47}$$

$$\mathcal{C}_n = \frac{1}{n^3}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{R_{x,i}}{n+1}\frac{R_{y,i}}{n+1} 1\{R_{x,k} \leq R_{x,i}, R_{x,k} \leq R_{x,j}\} + \frac{1}{4} - \mathcal{A}_n \tag{48}$$

$$\mathcal{D}_n = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{R_{y,i}}{n+1}\frac{R_{y,j}}{n+1} \max\left\{\frac{R_{x,i}}{n+1}, \frac{R_{x,j}}{n+1}\right\} \tag{49}$$

$$\mathcal{E}_n = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{R_{x,i}}{n+1}\frac{R_{x,j}}{n+1} \max\left\{\frac{R_{y,i}}{n+1}, \frac{R_{y,j}}{n+1}\right\}. \tag{50}$$

On the other hand, Kojadinovic and Yan (2010) also proposed a very simple but also consistent estimator of ξ^2 as

$$V_n^2 = \frac{144}{n-1}\sum_{i=1}^{n} (Z_i - \overline{Z})^2, \tag{51}$$

where

$$Z_i = \frac{R_{x,i}}{n+1}\frac{R_{y,i}}{n+1} + \frac{1}{n}\sum_{k=1}^{n} 1\{R_{x,i} \leq R_{x,k}\}\frac{R_{y,k}}{n+1} + \frac{1}{n}\sum_{k=1}^{n} 1\{R_{y,i} \leq R_{y,k}\}\frac{R_{x,k}}{n+1}, \tag{52}$$

$$\text{and } \overline{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_i. \tag{53}$$

Because $\xi_{\rho_s}^2$ is unknown in reality, in order to compute the required sample size, n_{SR}, we use sequential procedure. Our sequential stopping rule which helps find the estimate of the optimal sample size is as follows:

$$N_{SR} \text{ is the smallest integer } n(\geq m_{SR}) \text{ such that } \frac{4 z_{\alpha/2}^2}{\omega^2}\left(\hat{\xi}_{n,SR}^2 + n^{-1}\right), \tag{54}$$

where m_{SR} is the pilot sample which is same as the pilot sample size as defined in Equation 9.

We now find the characteristics of the stopping rule defined in Equation 54 using Monte Carlo simulation for constructing (1 − α)100% confidence intervals for population correlation coefficients from bivariate distributions — bivariate normal and the bivariate gamma distribution from Theorem 2 of Nadarajah and Gupta (2006). The simulation study was done for correlation coefficient τ corresponding to ρ = {0.1, 0.3, 0.5} and ω = {0.1, 0.2}. In all cases, 5,000 replications were used. Tables 9 and 10 show the estimates of mean final sample size, coverage probability, average confidence interval width, and standard error for 90% and 95% confidence intervals, respectively.

The width of the confidence interval given by the sequential procedure with stopping rule defined in Equation 54 did not exceed the maximum specified width ω. The coverage probability estimates are close to the corresponding confidence level. Also, the ratio of average sample size estimate and the optimal sample size is close to 1.

Table 9
Summary of Final Sample Size for 90% Confidence Interval for Spearman's Rho, ρ_s

ω	Distribution	ρ	ρ_s	$\bar{N}_{SR}$	$se(\bar{N}_{SR})$	n_{SR}	$\bar{N}_{SR}/n_{SR}$	p_{SR}	$se(p_{SR})$	$\bar{w}_{N_{SR}}$	$se(\bar{w}_{N_{SR}})$
.1	$N_2(0, 0, 1, 1, .1)$	.1	.0955	1,065	.4098	1,066	.9988	.8998	.0042	.0999	4.07×10^{-7}
	$N_2(0, 0, 1, 1, .3)$	.3	.2876	931.3	.6019	933	.9982	.8964	.0043	.0999	5.599×10^{-7}
	$N_2(0, 0, 1, 1, .5)$	.5	.4826	679.3	.8218	683	.9946	.8876	.0045	.0998	1.063×10^{-6}
	$Ga_2(5, 5, 50, 10)$	.1	.0915	1,069	.4031	1,116	.9579	.9002	.0042	.0999	3.943×10^{-7}
	$Ga_2(5, 5, 16.67, 10)$	.3	.2821	948.9	.6067	957	.9916	.9004	.0042	.0999	5.432×10^{-7}
	$Ga_2(5, 5, 10, 10)$	.5	.4765	714.2	.8437	695	1.0280	.8998	.0042	.0998	9.701×10^{-7}
.2	$N_2(0, 0, 1, 1, .1)$	.1	.0955	265.4	.2140	267	.9940	.8898	.0044	.1993	3.409×10^{-6}
	$N_2(0, 0, 1, 1, .3)$	.3	.2876	231.5	.3236	234	.9895	.8890	.0044	.1990	1.78×10^{-5}
	$N_2(0, 0, 1, 1, .5)$	.5	.4826	166.5	.4690	171	.9735	.8598	.0049	.1977	5.669×10^{-5}
	$Ga_2(5, 5, 50, 10)$	.1	.0915	266.2	.2102	279	.9542	.8924	.0044	.1993	3.444×10^{-6}
	$Ga_2(5, 5, 16.67, 10)$	.3	.2821	235.5	.3279	240	.9913	.8894	.0044	.1990	1.256×10^{-5}
	$Ga_2(5, 5, 10, 10)$	.5	.4765	174.9	.4832	174	1.0050	.8618	.0049	.1980	4.947×10^{-5}

Note. ρ is the population Pearson's correlation coefficient; ρ_s is the population Spearman's rho (computed using bootstrap method for bivariate Gamma distribution); $\bar{N}_{SR}$ is the mean final sample size; p_{SR} is the estimated coverage probability; ω is the upper bound of the length of the confidence interval for ρ_s; $se(\bar{N}_{SR})$ is the standard deviation of the mean final sample size (i.e., standard error of the final sample size); n_{SR} is the theoretical sample size if the procedure is used with the population parameters (computed using bootstrap method for bivariate Ga distribution); $se(p_{SR})$ is the standard error of p_{SR}; $\bar{w}_{N_{SR}}$ average length of confidence intervals for ρ_s based on N observations; tabled values are based on 5,000 replications of a Monte Carlo simulation study from distributions: bivariate normal (N_2) distribution with parameters μ_1, μ_2, σ_1^2, σ_2^2, and ρ, respectively, and bivariate gamma (Ga_2) with parameters a_1, a_2, c, and μ, respectively, based on Theorem 2 of Nadarajah and Gupta (2006).

One can proceed along the same lines as in this article (in Appendix B) and in Kelley et al. (2018) to prove that the respective coverage probabilities for the confidence interval for Kendall's τ (τ) and Spearman's ρ (ρ_s) are approximately close to the confidence level, and also the width of the confidence interval is less than ω.

An Extension: Squared Multiple Correlation Coefficient

The sequential procedure that is proposed for correlation coefficients in the previous sections is now extended for finding a sufficiently narrow confidence interval for the population squared multiple correlation coefficient. In this section, we develop the sequential AIPE procedure for the squared multiple correlation coefficient under multivariate normal assumption only. We first formulate the corresponding AIPE problem. Although the previous parts of the article were distribution-free, here we assume multivariate normality because there is not, to our knowledge, a sufficient analytic method for forming a confidence interval for the population squared multiple correlation coefficient that is distribution-free.

Suppose, for the $i^{\text{th}}(i = 1, 2, \ldots, n)$ individual out of n individuals, Y_i is the score corresponding to the response variable and X_{ij} is the observed score corresponding to the j^{th} ($j = 1,2, \ldots, k$) predictor variable. Let $\mathbf{Y}$ denote the random vector of responses

Table 10
Summary of Final Sample Size for 95% Confidence Interval for Spearman's Rho, ρ_s

ω	Distribution	ρ	ρ_s	$\bar{N}_{SR}$	$se(\bar{N}_{SR})$	n_{SR}	$\bar{N}_{SR}/n_{SR}$	p_{SR}	$se(p_{SR})$	$\bar{w}_{N_{SR}}$	$se(\bar{w}_{N_{SR}})$
.1	$N_2(0, 0, 1, 1, .1)$	.1	.0955	1,512	.4830	1,513	.9994	.9524	.0030	.0999	2.78×10^{-7}
	$N_2(0, 0, 1, 1, .3)$	.3	.2876	1,323	.7309	1,325	.9986	.9526	.0030	.0999	3.87×10^{-7}
	$N_2(0, 0, 1, 1, .5)$	.5	.4826	966.2	.9759	970	.9961	.9410	.0033	.0999	7.03×10^{-7}
	$Ga_2(5, 5, 50, 10)$	.1	.0915	1,518	.4826	1,584	.9583	.9530	.0030	.0999	2.77×10^{-7}
	$Ga_2(5, 5, 16.67, 10)$	.3	.2821	1,348	.7112	1,358	.9927	.9526	.0030	.0999	3.74×10^{-7}
	$Ga_2(5, 5, 10, 10)$	.5	.4765	1,015	.9828	986	1.0290	.9508	.0031	.0999	6.472×10^{-7}
.2	$N_2(0, 0, 1, 1, .1)$	.1	.0955	377.4	.2461	379	.9958	.9362	.0035	.1995	2.371×10^{-6}
	$N_2(0, 0, 1, 1, .3)$	.3	.2876	329.7	.3701	332	.9931	.9394	.0034	.1993	3.428×10^{-6}
	$N_2(0, 0, 1, 1, .5)$	.5	.4826	238.8	.5289	243	.9829	.9300	.0036	.1986	3.026×10^{-5}
	$Ga_2(5, 5, 50, 10)$	.1	.0915	378.7	.2450	396	.9562	.9478	.0031	.1995	2.352×10^{-6}
	$Ga_2(5, 5, 16.67, 10)$	.3	.2821	336.1	.3653	340	.9887	.9410	.0033	.1994	3.349×10^{-6}
	$Ga_2(5, 5, 10, 10)$	.5	.4765	250.9	.5369	247	1.0160	.9340	.0035	.1987	3.231×10^{-5}

Note. ρ is the population Pearson's correlation coefficient; ρ_s is the population Spearman's rho (computed using bootstrap method for bivariate Gamma distribution); $\bar{N}_{SR}$ is the mean final sample size; p_{SR} is the estimated coverage probability; ω is the upper bound of the length of the confidence interval for ρ_s; $se(\bar{N}_{SR})$ is the standard deviation of the mean final sample size (i.e., standard error of the final sample size); n_{SR} is the theoretical sample size if the procedure is used with the population parameters (computed using bootstrap method for bivariate Ga distribution); $se(p_{SR})$ is the standard error of p_{SR}; $\bar{w}_{N_{SR}}$ average length of confidence intervals for ρ_s based on N observations; tabled values are based on 5,000 replications of a Monte Carlo simulation study from distributions: bivariate normal (N_2) distribution with parameters μ_1, μ_2, σ_1^2, σ_2^2, and ρ, respectively, and bivariate gamma (Ga_2) with parameters a_1, a_2, c, and μ, respectively, based on Theorem 2 of Nadarajah and Gupta (2006).

Table 11
Summary of Final Sample Size for 90% Confidence Interval for P^2 Using Bonett and Wright (2011)

k	ω	P^2	$\bar{N}_{BW}$	$se(\bar{N}_{BW})$	n_{BW}	$\bar{N}_{BW}/n_{BW}$	p_{BW}	$se(p_{BW})$	$\bar{w}_{N_{BW}}$
2	.05	.10	1,308.44	5.4457	1,407	.9299	.8344	.0053	.0489
		.30	2,529.40	3.0025	2,551	.9915	.8866	.0045	.0499
		.50	2,149.81	2.6762	2,171	.9902	.8916	.0044	.0499
		.70	1,071.65	2.3025	1,098	.9760	.8886	.0044	.0496
		.90	140.63	.8780	164	.8575	.7894	.0058	.0481
	.10	.10	297.69	2.0105	355	.8386	.7404	.0062	.0959
		.30	623.99	1.3225	642	.9719	.8768	.0046	.0992
		.50	533.22	1.1170	547	.9748	.8858	.0045	.0992
		.70	262.79	.9621	280	.9385	.8554	.0050	.0982
		.90	34.84	.3222	47	.7412	.7688	.0060	.0908
5	.05	.10	1,414.03	2.8253	1,410	1.0029	.8948	.0043	.0498
		.30	2,525.96	3.5884	2,554	.9890	.8916	.0044	.0498
		.50	2,130.43	3.9169	2,174	.9800	.8846	.0045	.0497
		.70	1,056.26	2.8159	1,101	.9594	.8722	.0047	.0494
		.90	129.40	.9586	167	.7748	.7196	.0064	.0473
	.10	.10	364.23	1.3204	358	1.0174	.8802	.0046	.0989
		.30	630.03	1.2387	645	.9768	.8816	.0046	.0992
		.50	524.81	1.4068	550	.9542	.8716	.0047	.0987
		.70	252.39	1.1333	283	.8918	.8216	.0054	.0970
		.90	30.90	.3098	50	.6179	.6668	.0067	.0871
10	.05	.10	1,444.09	3.0292	1,415	1.0206	.8938	.0044	.0497
		.30	2,503.95	5.1902	2,559	.9785	.8908	.0044	.0496
		.50	2,101.63	5.2572	2,179	.9645	.8720	.0047	.0493
		.70	1,033.60	3.5467	1,106	.9345	.8510	.0050	.0489
		.90	113.23	1.0222	172	.6583	.6032	.0069	.0455
	.10	.10	391.00	1.3881	363	1.0771	.8580	.0049	.0981
		.30	621.40	1.8054	650	.9560	.8506	.0050	.0982
		.50	509.91	1.8875	555	.9188	.8298	.0053	.0973
		.70	234.17	1.3801	288	.8131	.7476	.0061	.0950
		.90	27.13	.2533	55	.4932	.5132	.0071	.0817

Note. P^2 is the population multiple correlation; $\bar{N}_{BW}$ is the mean final sample size; p_{BW} is the estimated coverage probability; ω is the upper bound of the length of the confidence interval for P^2; $se(\bar{N}_{BW})$ is the standard deviation of the mean final sample size (i.e., standard error of the final sample size); n_{BW} is the theoretical sample size if the procedure is used with known population value of P^2; $se(p_{BW})$ is the standard error of p_{BW}; $\bar{w}_{BW}$ average length of confidence intervals for P^2 based on N_{BW} observations; tabled values are based on 5,000 replications of a Monte Carlo simulation study from multivariate normal distribution (N_k) with parameters: mean vector μ and variance covariance matrix $\boldsymbol{\Sigma}$.

and $\mathbf{X}$ denote the corresponding random design matrix. The univariate linear regression model in matrix form is

$$\mathbf{Y} = \boldsymbol{\beta}_0\mathbf{1} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \tag{55}$$

where $\mathbf{1}$ is the vector where all elements are 1, $\boldsymbol{\beta}$ is the vector of regression parameters, and $\boldsymbol{\epsilon}$ is the random error vector. The population squared multiple correlation coefficient is given by

$$P^2 = \frac{\boldsymbol{\sigma}_{YX}\boldsymbol{\Sigma}_{XX}^{-1}\boldsymbol{\sigma}_{XY}}{\sigma_Y^2} \tag{56}$$

where $\boldsymbol{\Sigma}_{XX}^{-1}$ is the inverse of the $k \times k$ population covariance matrix of the k predictors, $\boldsymbol{\sigma}_{XY}$ is the k dimensional column vector of covariances of the k predictors with the response Y, $\boldsymbol{\sigma}_{YX}$ is the k dimensional row vector of covariances of the k predictors with the response Y ($\boldsymbol{\sigma}'_{XY} = \boldsymbol{\sigma}_{YX}$), and σ_Y^2 is the population variance of the response Y.

A well-known consistent estimator of the population squared multiple correlation coefficient, also known as R-squared (R^2) or multiple R-squared, is given by

$$R^2 = \frac{\mathbf{s}_{YX}\mathbf{S}_{XX}^{-1}\mathbf{s}_{XY}}{s_Y^2} \tag{57}$$

where $\mathbf{S}_{XX}^{-1}$ is the inverse of the $k \times k$ sample covariance matrix of the k predictors, $\mathbf{s}_{XY}$ is the k dimensional column vector of sample covariances of the k predictors with the response Y, $\mathbf{s}_{YX}$ is the k dimensional row vector of sample covariances of the k predictors with the response Y ($\mathbf{s}'_{XY} = \mathbf{s}_{YX}$), and s_Y^2 is the sample variance of the response Y.

The approximate $(1 - \alpha)100\%$ confidence interval for the population squared multiple correlation coefficient as given in Bonett and Wright (2011) is

$$1 - \exp\left(\ln(1 - R^2) \pm z_{\alpha/2}\frac{2P}{\sqrt{n-k-2}}\right), \tag{58}$$

where $z_{\alpha/2}$ is the $100(1-\alpha/2)^{\text{th}}$ percentile of the standard normal distribution. The approximate $(1 - \alpha)100\%$ Wald-type confidence interval for the population squared multiple correlation coefficient using the asymptotic variance developed by Olkin and Finn (1995) is given by

$$R^2 \pm z_{\alpha/2}\frac{2P(1 - P^2)}{\sqrt{n}}. \tag{59}$$

In AIPE for the population squared multiple correlation coefficient, in order to have a sufficiently narrow confidence interval, an upper bound (ω) of the confidence interval width is prespecified. Using the width constraint, we can find the optimal sample size for the confidence interval given in Equation 58 as

Table 12
Summary of Final Sample Size for 95% Confidence Interval for P^2 Using Bonett and Wright (2011)

k	ω	P^2	$\bar{N}_{BW}$	$se(\bar{N}_{BW})$	n_{BW}	$\bar{N}_{BW}/n_{BW}$	p_{BW}	$se(p_{BW})$	$\bar{w}_{N_{BW}}$
2	.05	.10	1,893.25	6.6567	1,996	.9485	.8966	.0043	.0492
		.30	3,596.31	3.8162	3,620	.9935	.9388	.0034	.0499
		.50	3,063.19	2.8966	3,080	.9945	.9448	.0032	.0499
		.70	1,537.45	2.3780	1,557	.9874	.9410	.0033	.0498
		.90	212.23	.9910	230	.9228	.8940	.0044	.0490
	.10	.10	435.52	2.5998	503	.8659	.8132	.0055	.0968
		.30	892.21	1.5674	910	.9805	.9374	.0034	.0995
		.50	760.73	1.3081	775	.9816	.9386	.0034	.0995
		.70	380.06	1.0911	395	.9622	.9254	.0037	.0989
		.90	50.55	.4177	64	.7899	.8432	.0051	.0936
5	.05	.10	2,005.04	3.3728	1,999	1.0030	.9432	.0033	.0498
		.30	3,600.12	3.8252	3,623	.9937	.9486	.0031	.0499
		.50	3,055.20	3.5829	3,083	.9910	.9432	.0033	.0499
		.70	1,520.37	3.2019	1,560	.9746	.9322	.0036	.0496
		.90	198.56	1.1441	233	.8522	.8384	.0052	.0482
	.10	.10	513.15	1.5693	506	1.0141	.9334	.0035	.0993
		.30	897.13	1.5073	913	.9826	.9342	.0035	.0993
		.50	754.62	1.6021	778	.9699	.9284	.0036	.0991
		.70	369.88	1.2873	398	.9294	.9074	.0041	.0982
		.90	45.27	.4144	67	.6757	.7616	.0060	.0910
10	.05	.10	2,034.40	3.6812	2,004	1.0152	.9446	.0032	.0498
		.30	3,588.62	5.2427	3,628	.9891	.9446	.0032	.0498
		.50	3,028.29	5.4085	3,088	.9807	.9402	.0034	.0497
		.70	1,499.93	4.0235	1,565	.9584	.9228	.0038	.0493
		.90	182.09	1.2902	238	.7651	.7526	.0061	.0473
	.10	.10	541.94	1.5947	511	1.0606	.9308	.0036	.0990
		.30	893.74	1.9830	918	.9736	.9236	.0038	.0990
		.50	744.97	2.0460	783	.9514	.9100	.0040	.0985
		.70	354.90	1.6083	403	.8806	.8572	.0049	.0969
		.90	39.85	.3800	72	.5534	.6460	.0068	.0879

Note. P^2 is the population multiple correlation; $\bar{N}_{BW}$ is the mean final sample size; p_{BW} is the estimated coverage probability; ω is the upper bound of the length of the confidence interval for P^2; $se(\bar{N}_{BW})$ is the standard deviation of the mean final sample size (i.e., standard error of the final sample size); n_{BW} is the theoretical sample size if the procedure is used with known population value of P^2; $se(p_{BW})$ is the standard error of p_{BW}; $\bar{w}_{BW}$ average length of confidence intervals for P^2 based on N_{BW} observations; tabled values are based on 5,000 replications of a Monte Carlo simulation study from multivariate normal distribution with parameters: mean vector μ and variance covariance matrix **Σ**.

$$n \geq 2 + k + 4P^2 z_{\alpha/2}^2 \left\{ \ln\left[\frac{1}{2}\left(\frac{\omega}{(1-P^2)} + \sqrt{\frac{\omega^2}{(1-P^2)^2} + 4} \right) \right] \right\}^{-2} \equiv n_{BW}, \quad (60)$$

and the optimal sample size for the confidence interval given in Equation 59 as

$$n \geq \frac{16 z_{\alpha/2}^2}{\omega^2} (P^2(1-P^2)^2) \equiv n_{OF}. \quad (61)$$

The derivation of the optimal sample sizes is given in Appendix C. Thus, n_{BW} is the optimal sample size which is required to get a sufficiently narrow confidence interval, of the form given in Equation 58, of the multiple correlation coefficient P^2. n_{OF} is the optimal sample size which is required to obtain a sufficiently narrow confidence interval of the form given in Equation 59 of the multiple correlation coefficient P^2. Note also that Kelley (2008) provided an AIPE procedure for the population squared multiple correlation coefficient using noncentral distributions (e.g., Kelley, 2007a). The method in this article differs from Kelley (2008) in that it is based on Olkin and Finn (1995) method instead of Steiger & Fouladi (1992, 1997).

Because P^2 is unknown, both n_{BW} and n_{OF} are also unknown. Thus, in order to obtain a sufficiently narrow confidence interval for P^2, we need to estimate n_{BW} and n_{OF}. This can be done using sequential procedure similar to what was described earlier.

The stopping rule related to the sequential procedure for estimating n_{bw} is given by:

N_{BW} is the smallest integer $n(\geq m_{BW})$ such that

$$n \geq 2 + k + 4R^2 z_{\alpha/2}^2 \left\{ \ln\left[\frac{1}{2}\left(\frac{\omega}{(1-R^2)} + \sqrt{\frac{\omega^2}{(1-R^2)^2} + 4} \right) \right] \right\}^{-2}. \quad (62)$$

We propose the corresponding pilot sample size to be $m_{BW} = k + 2$. Now, the stopping rule related to the sequential procedure for estimating n_{OF} is given by:

N_{OF} is the smallest integer $n(\geq m_{OF})$ such that

$$n \geq \frac{16 z_{\alpha/2}^2}{\omega^2} \left[\left(R^2 + \frac{1}{n}\right)\left(1 - \left(R^2 + \frac{1}{n}\right)\right)^2 \right], \quad (63)$$

where $m_{OF} = \max\{k, 4z_{\alpha/2}/\omega\}$ is the corresponding pilot sample size. We note that, in Equation 63, we use $R^2 + 1/n$ which is a consistent estimator of P^2. The expression of the pilot sample sizes m_{BW} and m_{OF} is derived in Appendix C.

Table 13
Summary of Final Sample Size for 90% Confidence Interval for P^2 Using Olkin and Finn (1995)

k	ω	P^2	$\bar{N}_{OF}$	$se(\bar{N}_{OF})$	n_{OF}	$\bar{N}_{OF}/n_{OF}$	p_{OF}	$se(p_{OF})$	$\bar{w}_{N_{OF}}$
2	.05	.10	1,406.47	2.4778	1,403	1.0025	.8902	.0044	.0497
		.30	2,548.32	.2678	2,546	1.0009	.8986	.0043	.0499
		.50	2,165.16	.9286	2,165	1.0001	.9024	.0042	.0499
		.70	1,091.74	1.2002	1,091	1.0007	.9026	.0042	.0498
		.90	174.44	.4987	156	1.1182	.8898	.0044	.0461
	.10	.10	347.98	1.3031	351	.9914	.8460	.0051	.0962
		.30	638.92	.1593	637	1.0030	.8930	.0044	.0995
		.50	543.16	.4694	542	1.0021	.8962	.0043	.0993
		.70	275.69	.6141	273	1.0099	.8830	.0045	.0981
		.90	68.66	.0963	39	1.7605	.8678	.0048	.0733
5	.05	.10	1,427.36	2.3615	1,403	1.0174	.8900	.0044	.0497
		.30	2,549.42	.2636	2,546	1.0013	.8938	.0044	.0499
		.50	2,161.49	.9365	2,165	.9984	.8998	.0042	.0499
		.70	1,087.11	1.2317	1,091	.9964	.8908	.0044	.0498
		.90	170.13	.4898	156	1.0906	.8684	.0048	.0458
	.10	.10	375.77	1.0788	351	1.0706	.8888	.0044	.0976
		.30	640.31	.1399	637	1.0052	.8932	.0044	.0995
		.50	539.22	.4790	542	.9949	.8854	.0045	.0993
		.70	268.82	.6312	273	.9847	.8604	.0049	.0980
		.90	67.94	.0812	39	1.7420	.8406	.0052	.0710
10	.05	.10	1,458.75	2.2223	1,403	1.0397	.8964	.0043	.0497
		.30	2,550.97	.2510	2,546	1.0020	.9016	.0042	.0499
		.50	2,157.60	.9383	2,165	.9966	.8962	.0043	.0499
		.70	1,080.75	1.2345	1,091	.9906	.8898	.0044	.0498
		.90	163.12	.4639	156	1.0456	.8106	.0055	.0452
	.10	.10	403.12	.9458	351	1.1485	.8736	.0047	.0981
		.30	641.16	.1292	637	1.0065	.8820	.0046	.0995
		.50	534.97	.4924	542	.9870	.8736	.0047	.0993
		.70	261.45	.6568	273	.9577	.8236	.0054	.0979
		.90	67.00	.0584	39	1.7179	.7294	.0063	.0661

Note. P^2 is the population multiple correlation coefficient; $\bar{N}_{OF}$ is the mean final sample size; p_{OF} is the estimated coverage probability; ω is the upper bound of the length of the confidence interval for P^2; $se(\bar{N}_{OF})$ is the standard deviation of the mean final sample size (i.e., standard error of the final sample size); n_{OF} is the theoretical sample size if the procedure is used with known population value of P^2; $se(p_{OF})$ is the standard error of p_{OF}; $\bar{w}_{N_{OF}}$ average length of confidence intervals for P^2 based on N_{OF} observations; tabled values are based on 5,000 replications of a Monte Carlo simulation study from multivariate normal distribution with parameters: mean vector μ and variance covariance matrix $\boldsymbol{\Sigma}$.

Simulation Results

We now find the characteristics of the stopping rules defined in Equations 62 and 63 using Monte Carlo simulation for constructing $(1 - \alpha)100\%$ confidence intervals for population squared multiple correlation coefficient from multivariate normal distributions with mean parameter vector and dispersion matrix respectively given by

$$\mu = (0, \ldots, 0)' = \mathbf{0}_{(k+1) \times 1}$$

and

$$\boldsymbol{\Sigma}_{(k+1) \times (k+1)} = \begin{bmatrix} 1 & \gamma' \\ \gamma & \mathbf{I} \end{bmatrix}$$

where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)' = \boldsymbol{\sigma}_{\mathbf{YX}}$ with $\gamma_i = \sqrt{P^2/k}$ for $i = 1, \ldots, k$, and $\mathbf{I}$ is a $k \times k$ identity matrix (i.e., $\text{cov}(X) = \mathbf{I}$). The simulation study was done for population squared multiple correlation under several scenarios with replication size 5,000. Tables 11, 12, 13, and 14 show the estimates of mean final sample size, coverage probability, and also the corresponding standard errors and the average confidence interval width. The simulation results show that the width of the confidence interval given by the sequential procedure with stopping rules defined in Equations 62–63 did not exceed the maximum specified width ω. Except for smaller optimal sample sizes, the coverage probability estimates are close to the corresponding confidence level and also, the ratio of average sample size estimate and the optimal sample size is close to 1. Also, we note that for relatively smaller sample sizes, the sequential procedure under both Olkin and Finn (1995) and Bonett and Wright (2011) confidence intervals produce estimated sample sizes and coverage probabilities that are comparatively lower than their respective targets. Overall, the results for the sequential procedure corresponding to the confidence interval given by Olkin and Finn (1995) performed better than the procedure given by Bonett and Wright (2011) in terms of required optimal sample size. Because of the analytic complexities associated with noncentral distributions, we did not develop the sequential procedures for the noncentral confidence interval approaches. However, in principle, the sequential AIPE procedure should generalize to such situations.

Discussion

A variety of correlation coefficients exist and are used in a wide variety of context in psychology and related fields. Estimating the population value of correlation coefficient is of great importance.

Table 14
Summary of Final Sample Size for 95% Confidence Interval for P^2 Using Olkin and Finn (1995)

k	ω	P^2	$\bar{N}_{OF}$	$se(\bar{N}_{OF})$	n_{OF}	$\bar{N}_{OF}/n_{OF}$	p_{OF}	$se(p_{OF})$	$\bar{w}_{N_{OF}}$
2	.05	.10	1,997.80	2.9059	1,992	1.0029	.9404	.0033	.0498
		.30	3,616.98	.3207	3,615	1.0005	.9432	.0033	.0500
		.50	3,074.28	1.1125	3,074	1.0001	.9498	.0031	.0499
		.70	1,550.08	1.4774	1,549	1.0007	.9412	.0033	.0498
		.90	237.06	.6750	222	1.0679	.9270	.0037	.0473
	.10	.10	495.01	1.5851	498	.9940	.9152	.0039	.0976
		.30	906.47	.1733	904	1.0027	.9474	.0032	.0996
		.50	769.43	.5548	769	1.0006	.9522	.0030	.0995
		.70	389.45	.7446	388	1.0037	.9328	.0035	.0987
		.90	84.17	.1493	56	1.5030	.9222	.0038	.0790
5	.05	.10	2,016.71	2.8057	1,992	1.0124	.9492	.0031	.0498
		.30	3,618.07	.3129	3,615	1.0008	.9504	.0031	.0500
		.50	3,070.70	1.1156	3,074	.9989	.9502	.0031	.0499
		.70	1,544.58	1.4603	1,549	.9971	.9492	.0031	.0498
		.90	230.38	.6648	222	1.0377	.9108	.0040	.0471
	.10	.10	522.35	1.3550	498	1.0489	.9442	.0032	.0983
		.30	907.40	.1615	904	1.0038	.9442	.0032	.0996
		.50	766.17	.5611	769	.9963	.9450	.0032	.0995
		.70	383.90	.7328	388	.9894	.9320	.0036	.0986
		.90	82.94	.1290	56	1.4810	.8988	.0043	.0769
10	.05	.10	2,047.95	2.6772	1,992	1.0281	.9508	.0031	.0498
		.30	3,619.31	.3047	3,615	1.0012	.9522	.0030	.0500
		.50	3066.39	1.0971	3,074	.9975	.9538	.0030	.0499
		.70	1,540.85	1.4559	1,549	.9947	.9466	.0032	.0498
		.90	223.07	.6671	222	1.0048	.8720	.0047	.0469
	.10	.10	552.43	1.1856	498	1.1093	.9406	.0033	.0986
		.30	908.41	.1527	904	1.0049	.9384	.0034	.0996
		.50	762.34	.5808	769	.9913	.9360	.0035	.0995
		.70	376.14	.7833	388	.9694	.9038	.0042	.0986
		.90	81.28	.0972	56	1.4513	.8304	.0053	.0728

Note. P^2 is the population multiple correlation coefficient; $\bar{N}_{OF}$ is the mean final sample size; p_{OF} is the estimated coverage probability; ω is the upper bound of the length of the confidence interval for P^2; $se(\bar{N}_{OF})$ is the standard deviation of the mean final sample size (i.e., standard error of the final sample size); n_{OF} is the theoretical sample size if the procedure is used with known population value of P^2; $se(p_{OF})$ is the standard error of p_{OF}; $\bar{w}_{N_{OF}}$ average length of confidence intervals for P^2 based on N_{OF} observations; tabled values are based on 5,000 replications of a Monte Carlo simulation study from multivariate normal distribution with parameters: mean vector μ and variance covariance matrix $\mathbf{\Sigma}$.

The necessity of using a $(1 - \alpha)100\%$ confidence interval that brackets a wide range of values in order to include the true value, with the specified level of confidence, represents an important problem. Correspondingly, a method to obtain a sufficiently narrow $(1 - \alpha)100\%$ confidence interval for the population correlation coefficient with a confidence interval width no larger than desired is very advantageous in many research contexts. However, until now, all such approaches required the specification of unknown population values and bivariate normality for Pearson product–moment correlation coefficient, Spearman's rho, and Kendall's tau. Our approach overcomes both of these limitations for these three important correlation coefficients. We discuss a distribution-free confidence interval approach for the population correlation coefficients, namely Pearson's product-moment correlation coefficient, Kendall τ rank correlation coefficient, and Spearman's ρ rank correlation coefficient. We then use distribution-free framework to develop a sequential approach to accuracy in parameter estimation of the correlation coefficients.

It is known that, holding constant the population value and confidence coefficient, a narrower confidence interval provides more information about the parameter than a wider confidence interval. Given a value of the upper bound of the confidence interval (ω), an approximate $(1 - \alpha)100\%$ confidence interval for the population correlation coefficient can be constructed by using an a priori sample size planning approach, which requires a supposed value of the population parameter. Using supposed population values based on theory, an estimate from one or more other studies, or a conjecture based on a rule of thumb can lead to sample size estimates that grossly differ from what the theoretically optimal sample size would be if the population parameters were known and assumptions were satisfied. We overcome such a limitation by proposing a sequential procedure which can be used to construct an approximate $(1 - \alpha)100\%$ confidence interval for the population correlation coefficients within a prespecified width (ω) without assuming any distribution of the data. Unlike a priori sample size planning approaches, our sequential procedure does not require knowledge of population parameters in order to obtain a sufficiently narrow confidence interval.

We discuss a sequential approach to construct a sufficiently narrow confidence interval for the population correlation coefficients of interest assuming homogeneity of the data distribution. Some studies (e.g., Stanley, Wilson, & Milfont, 2017; van Erp, Verhagen, Grasman, & Wagenmakers, 2017) have shown that heterogeneity of the data distribution is possible, due to which the population effect size may change (e.g., parameter drift). However, incorporating heterogeneity or parameter drift is beyond the scope of the article. To the extent that heterogeneity or parameter drift exists over the time frame in which data are collected, it would be a limitation. Additionally, the methods we use for confidence interval construction do not work well in all situations, particularly for small sample sizes combined with non-normal data. Nevertheless, without assuming any particular distribution, the methods we use for Pearson's, Spearman's, and Kendall's correlations are distribution-free and, provided sample size is not too small for the particular situation, will work well as sample size gets larger. See Chattopadhyay and Kelley (2016) for a discussion of distribution free limitation, particularly Footnote 9.

We also develop sequential AIPE for obtaining a sufficiently narrow confidence interval for population squared multiple correlation coefficient. Without assuming the population value (P^2), the method provide a procedure to obtain the smallest possible size to obtain a $(1 - \alpha)100\%$ confidence interval with a desirable width using either Bonett and Wright (2011) or Olkin and Finn (1995). Another limitation in this regard is that our sequential methods for the squared multiple correlation coefficient require multivariate normality, as there is not a well developed distribution-free confidence interval method for the squared multiple correlation coefficient. Besides this, the procedure works well as sample size increases but not for small sample sizes.

As a general overview of our procedure, we first obtain a pilot sample size. After collecting the pilot data, we then use a sequential sampling procedure where, at each stage, we check whether a stopping rule has been satisfied. If not, additional observation(s) from one or more individuals, depending on the selected sample size at each stage, on both variables are collected and the check is performed again. This process continues until the stopping rule is satisfied. Our method ensures that the length of the confidence interval for correlation coefficient is less than the desired width and also attains the coverage probability asymptotically while using the smallest possible sample size. Based on the limitation of existing sample size procedures with regard to distribution assumption and assumed knowledge of population parameters, our sequential procedure has the potential to be widely used in psychology and related disciplines. To help researchers, we have provided freely available R functions via the MBESS package.

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(*Appendices follow*)

Appendix A

A Consistent Estimator for the Asymptotic Variance of the Pearson's Product Moment Correlation Coefficient

Using A. J. Lee (1990), the estimator of the covariance of X and Y, denoted σ_{XY}, is given in U-statistics form as

$$S_{XYn} \equiv U_{1n} = \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} \frac{1}{2}(X_i - X_j)(Y_i - Y_j), \tag{64}$$

where we use a subscript n to denote the current sample size used in the estimation. The expression of the sample covariance $S_{XY\,n}$ in Equation 64 is similar to

$$S_{XYn} = (n-1)^{-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)(Y_i - \bar{Y}_n). \tag{65}$$

The unbiased estimators of the variance parameters σ_X^2 and σ_Y^2 are, respectively,

$$S_{Xn}^2 \equiv U_{2n} = \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} \frac{1}{2}(X_i - X_j)^2 \tag{66}$$

and

$$S_{Yn}^2 \equiv U_{3n} = \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} \frac{1}{2}(Y_i - Y_j)^2. \tag{67}$$

The expression of the sample variances S_{Xn}^2 and S_{Yn}^2 in Equations 66–67 is similar to

$$S_{Xn}^2 = (n-1)^{-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \tag{68}$$

and

$$S_{Yn}^2 = (n-1)^{-1} \sum_{i=1}^{n} (Y_i - \bar{Y}_n)^2, \tag{69}$$

respectively. For technical details regarding the equivalence, we refer the reader to A. J. Lee (1990) and Mukhopadhyay and Chattopadhyay (2013, 2014). The U-statistics form of the expressions of sample covariance and sample variances are required for proving Theorem 1 in Appendix B.

Consistent Estimator of ξ^2

We note that S_{XYn}, S_{Xn}^2, and S_{Yn}^2 defined in Equations 64–67 are U-statistics or unbiased statistics and are consistent estimators of σ_{XY}, σ_X^2, and σ_Y^2, respectively. For details regarding U-statistics and their properties, we refer to Hoeffding (1948, 1961) and Lee (1990).

Using Lee (1990), a consistent estimator of ξ^2 is $\hat{\xi}_n^2 = \max\{V_n^2, n^{-3}\}$, where V_n^2 is

$$V_n^2 = \frac{r_n^2}{4}\left(\frac{\hat{\mu}_{40n}}{S_{Xn}^4} + \frac{\hat{\mu}_{04n}}{S_{Yn}^4} + \frac{2\hat{\mu}_{22n}}{S_{Xn}^2 S_{Yn}^2} + \frac{4\hat{\mu}_{22n}}{S_{XYn}^2} - \frac{4\hat{\mu}_{31n}}{S_{XYn}S_{Xn}^2} - \frac{4\hat{\mu}_{13n}}{S_{XYn}S_{Yn}^2}\right). \tag{70}$$

$\hat{\mu}_{40n}$ and $\hat{\mu}_{04n}$ are the respective unbiased estimators of the fourth central moment of X (μ_{40}) and Y (μ_{04}) which are given respectively as:

$$\begin{aligned} \hat{\mu}_{40n} = & \frac{n^2}{(n-1)(n-2)(n-3)} \sum_{i=1}^{n} (X_i - \bar{X}_n)^4 \\ & - \frac{2n-3}{(n-1)(n-2)(n-3)} \sum_{i=1}^{n} X_i^4 \\ & + \frac{8n-12}{(n-1)(n-2)(n-3)} \bar{X}_n \sum_{i=1}^{n} X_i^3 \\ & - \frac{6n-9}{n(n-1)(n-2)(n-3)} \left(\sum_{i=1}^{n} X_i^2\right)^2 \end{aligned} \tag{71}$$

and

$$\begin{aligned} \hat{\mu}_{04n} = & \frac{n^2}{(n-1)(n-2)(n-3)} \sum_{i=1}^{n} (Y_i - \bar{Y}_n)^4 \\ & - \frac{2n-3}{(n-1)(n-2)(n-3)} \sum_{i=1}^{n} Y_i^4 \\ & + \frac{8n-12}{(n-1)(n-2)(n-3)} \bar{Y}_n \sum_{i=1}^{n} Y_i^3 \\ & - \frac{6n-9}{n(n-1)(n-2)(n-3)} \left(\sum_{i=1}^{n} Y_i^2\right)^2. \end{aligned} \tag{72}$$

According to Cook (1951) and R. A. Fisher (1930), the remaining estimators can be defined as

$$\begin{aligned} \hat{\mu}_{13n} &= k_{13} + 3k_{02}k_{11} \\ \hat{\mu}_{22n} &= k_{22} + k_{20}k_{02} + 2k_{11}^2 \\ \hat{\mu}_{31n} &= k_{31} + 3k_{20}k_{11} \end{aligned} \tag{73}$$

where

(*Appendices continue*)

$$
\begin{aligned}
W_{pq} &= \sum_i X_i^p Y_i^q \\
k_{02} &= \frac{1}{n-1}\left(W_{02} - \frac{1}{n}W_{01}^2\right) = S_{Yn}^2 \\
k_{11} &= \frac{1}{n-1}\left(W_{11} - \frac{1}{n}W_{10}W_{01}\right) = S_{XYn} \\
k_{13} &= \frac{n}{(n-1)(n-2)(n-3)}\Big\{(n+1)W_{13} - \frac{n+1}{n}W_{03}W_{10} - \frac{3(n-1)}{n}W_{11}W_{02} \\
&\quad - \frac{3(n+1)}{n}W_{12}W_{01} + \frac{6}{n}W_{11}W_{01}^2 + \frac{6}{n}W_{02}W_{01}W_{10} - \frac{6}{n^2}W_{10}W_{01}^3\Big\} \\
k_{20} &= \frac{1}{n-1}\left(W_{20} - \frac{1}{n}W_{10}^2\right) = S_{Xn}^2 \\
k_{22} &= \frac{n}{(n-1)(n-2)(n-3)}\Big\{(n+1)W_{22} - \frac{2(n+1)}{n}W_{21}W_{01} - \frac{n+1}{n}W_{12}W_{10} \\
&\quad - \frac{n-1}{n}W_{20}W_{02} - \frac{2(n-1)}{n}W_{11}^2 + \frac{8}{n}W_{11}W_{01}W_{10} + \frac{2}{n}W_{02}W_{10}^2 + \frac{2}{n}W_{20}W_{01}^2 - \frac{6}{n^2}W_{10}^2W_{01}^2\Big\} \\
k_{31} &= \frac{n}{(n-1)(n-2)(n-3)}\Big\{(n+1)W_{31} - \frac{n+1}{n}W_{30}W_{01} - \frac{3(n-1)}{n}W_{11}W_{20} \\
&\quad - \frac{3(n+1)}{n}W_{21}W_{10} + \frac{6}{n}W_{11}W_{10}^2 + \frac{6}{n}W_{20}W_{10}W_{01} - \frac{6}{n^2}W_{01}W_{10}^3\Big\}
\end{aligned}
\tag{74}
$$

Appendix B

Lemmas and Theorems for Stopping Rules

Lemmas and Theorems

Lemma 1

Under the assumption that $E[\hat{\xi}_n^2]$ exists, for any $\omega > 0$, the stopping rule N is finite, that is, $P(N < \infty) = 1$.

Proof

Using Lemma A1 of Chattopadhyay and De (2016), we can prove the lemma.

Lemma 2

If the parent distribution(s) is(are) such that $E[\hat{\xi}_n^2]$ exists, then the stopping rule in Equation 10 yields

$$\frac{N}{n_\omega} \xrightarrow{P} 1 \text{ as } \omega \to 0, \tag{75}$$

where $\xrightarrow{P}$ indicates convergence in probability.

Proof

To prove the lemma, we proceed along the lines of Chattopadhyay and Kelley (2017; see also De & Chattopadhyay, 2017). The definition of stopping rule N in Equation 10 yields

$$\left(\frac{2z_{\alpha/2}}{\omega}\right)^2 \hat{\xi}_N^2 \le N \le mI(N=m) + \left(\frac{2z_{\alpha/2}}{\omega}\right)^2 \left(\hat{\xi}_{N-1}^2 + (N-1)^{-1}\right). \tag{76}$$

Because $N \to \infty$ asymptotically as $\omega \downarrow 0$ and $\hat{\xi}_n \to \xi$ in probability as $n \to \infty$, by Theorem 2.1 of Gut (2009), $\hat{\xi}_N^2 \to \xi^2$ in probability. Hence, dividing all sides of Equation 76 by n_ω and letting $\omega \downarrow 0$, we prove $N/n_\omega \to 1$ asymptotically as $\omega \downarrow 0$.

Theorem 1

Suppose the parent distribution F is such that $E[U_{in}^2] < \infty$ for $i = 1, 2, 3$, then the stopping rule in Equation 10 yields:

$$
\begin{aligned}
&\text{Part 1: } P\left(r_N - \frac{z_{\alpha/2}\hat{\xi}_N}{\sqrt{N}} < \rho < r_N + \frac{z_{\alpha/2}\hat{\xi}_N}{\sqrt{N}}\right) \to 1-\alpha \quad \text{as} \quad \omega \to 0, \\
&\text{Part 2: } \frac{2z_{\alpha/2}\hat{\xi}_N}{\sqrt{N}} \le \omega.
\end{aligned}
\tag{77}
$$

Proof

Part 1: We now proceed along the lines of De and Chattopadhyay (2017). Let $\mathbf{U}_n = [U_{1n}, U_{2n}, U_{3n}]'$ and $\theta = [\sigma_{XY}, \sigma_X^2, \sigma_Y^2]'$, then from A. J. Lee (1990), we know that

$$\mathbf{Y}_n = \sqrt{n}[\mathbf{U}_n - \theta] \xrightarrow{\mathcal{L}} N_3(\mathbf{0}, \boldsymbol{\Sigma}), \tag{78}$$

where

$$\boldsymbol{\Sigma} = \begin{bmatrix} \mu_{22} - \sigma_{XY}^2 & \mu_{31} - \sigma_{XY}\sigma_X^2 & \mu_{13} - \sigma_{XY}\sigma_Y^2 \\ \mu_{31} - \sigma_{XY}\sigma_X^2 & \mu_{40} - \sigma_X^4 & \mu_{22} - \sigma_X^2\sigma_Y^2 \\ \mu_{13} - \sigma_{XY}\sigma_Y^2 & \mu_{22} - \sigma_X^2\sigma_Y^2 & \mu_{04} - \sigma_Y^4 \end{bmatrix}.$$

(*Appendices continue*)

We define $\mathbf{D}' = [a_1, a_2, a_3]$ and note that $\mathbf{D}'\mathbf{Y}_N = \mathbf{D}'\mathbf{Y}_{n_\omega} + (\mathbf{D}'\mathbf{Y}_N - \mathbf{D}'\mathbf{Y}_{n_\omega})$. To prove that $\mathbf{Y}_N \xrightarrow{\mathcal{L}} N_3(\mathbf{0}, \boldsymbol{\Sigma})$, we have to show that $\mathbf{D}'(\mathbf{Y}_N - \mathbf{Y}_{n_\omega}) \xrightarrow{P} 0$ as $\omega \to 0$. We write

$$\mathbf{D}'(\mathbf{Y}_N - \mathbf{Y}_{n_\omega}) = \sum_{i=1}^{3} a_i \sqrt{N}(U_{iN} - U_{in_\omega}) + (\sqrt{N/n_\omega} - 1)\mathbf{D}'\mathbf{Y}_{n_\omega}. \tag{79}$$

Let $n_1 = (1 - \gamma)n_\omega$ and $n_2 = (1 + \gamma)n_\omega$ for $\gamma \in (0, 1)$. For a Fixed $\varepsilon > 0$,

$$\begin{aligned} P&\left\{ \left| \sum_{i=1}^{3} a_i \sqrt{N}\left(U_{iN} - U_{in_\omega}\right) \right| > \varepsilon \right\} \\ \le P&\left\{ \left| \sum_{i=1}^{3} a_i \sqrt{N}\left(U_{iN} - U_{in_\omega}\right) \right| > \varepsilon, |N - n_\omega| < \gamma n_\omega \right\} \\ &+ P\{|N - n_\omega| > \gamma n_\omega\} \\ \le &\sum_{i=1}^{3} P\left\{ \max_{n_1 < n < n_2} \sqrt{n}|U_{in} - U_{in_\omega}| > \frac{\varepsilon}{3|a_i|} \right\} \\ &+ P\{|N - n_\omega| > \gamma n_\omega\}. \end{aligned} \tag{80}$$

Because $N/n_\omega \xrightarrow{P} 1$ and U_{in}, $i = 1, 2, 3$ are U-statistics which satisfy Anscombe's uniformly continuous in probability condition (see Anscombe, 1952), then we conclude that $\sum_{i=1}^{3} a_i \sqrt{N}(U_{iN} - U_{in_\omega}) \xrightarrow{P} 0$. Also, $(\sqrt{N/n_\omega} - 1)\mathbf{D}'\mathbf{Y}_{n_\omega} \xrightarrow{P} 0$ as $\omega \to 0$ and $\mathbf{D}'\mathbf{Y}_{n_\omega} \xrightarrow{\mathcal{L}} N(\mathbf{0}, \mathbf{D}'\boldsymbol{\Sigma}\mathbf{D})$. Hence, we conclude from Equation 79 that $\mathbf{D}'(\mathbf{Y}_N - \mathbf{Y}_{n_\omega}) \xrightarrow{P} 0$, that is, $\mathbf{Y}_N \xrightarrow{\mathcal{L}} N_3(\mathbf{0}, \boldsymbol{\Sigma})$. Now, we define $g(u_1, u_2, u_3) = \frac{u_1}{\sqrt{u_2 u_3}}$ for $u_2, u_3 > 0$ and rewrite $r_n = g(\mathbf{U}_n)$ using Taylor's expansion about θ:

$$\begin{aligned} g(\mathbf{U}_n) = g(\theta) &+ \frac{U_{1N} - \sigma_{XY}}{\sigma_X \sigma_Y} - \frac{\sigma_{XY}}{2\sigma_X^3 \sigma_Y}(U_{2N} - \sigma_X^2) \\ &- \frac{\sigma_{XY}}{2\sigma_X \sigma_Y^3}(U_{3N} - \sigma_Y^2) + R_N, \end{aligned} \tag{81}$$

where

$$R_N = \frac{1}{2}(\mathbf{U}_N - \theta)'\{D^2 g(\mathbf{a})\}(\mathbf{U}_N - \theta) \tag{82}$$

and $D^2 g(\mathbf{a})$ is the Hessian matrix of $g(\mathbf{U}_n)$ evaluated at $\mathbf{a} = (1 - \gamma)\theta + \gamma \mathbf{U}_n$ for $\gamma \in (0, 1)$. Thus,

$$\begin{aligned} \sqrt{N}(r_N - \rho) &= \sqrt{N}\frac{\rho}{2}\left(\frac{2}{\sigma_{XY}}(U_{1N} - \sigma_{XY}) - \frac{1}{\sigma_X^2}(U_{2N} - \sigma_X^2) \right. \\ &\qquad \left. - \frac{1}{\sigma_Y^2}(U_{3,N} - \sigma_3^2) \right) + \sqrt{N}R_N \\ &= \mathbf{D}'\, \mathbf{Y}_N + \sqrt{N}R_N \end{aligned} \tag{83}$$

where $\rho = g(\boldsymbol{\theta})$ and $\mathbf{D}' = \frac{\rho}{2}\left[\frac{2}{\sigma_{XY}}, -\frac{1}{\sigma_X^2}, -\frac{1}{\sigma_Y^2}\right]$.

According to Lee (1990) and Anscombe's CLT (see Anscombe, 1952), $\sqrt{N}(U_{1N} - \sigma_{XY})$, $\sqrt{N}(U_{2N} - \sigma_X^2)$, and $\sqrt{N}(U_{3N} - \sigma_Y^2)$ converge to normal distributions and $(U_{1N} - \sigma_{XY})$, $(U_{2N} - \sigma_X^2)$, and $(U_{3N} - \sigma_Y^2)$ converge to 0 almost surely. This implies $\sqrt{N}R_N \xrightarrow{P} 0$ as $N \to \infty$. Hence, $\sqrt{N}(r_N - \rho) \xrightarrow{\mathcal{L}} N(0, \xi^2)$ as $\omega \to 0$, where

$$\xi^2 = \mathbf{D}'\boldsymbol{\Sigma}\mathbf{D} = \frac{\rho^2}{4}\left(\frac{\mu_{40}}{\sigma_X^4} + \frac{\mu_{04}}{\sigma_Y^4} + \frac{2\mu_{22}}{\sigma_X^2\sigma_Y^2} + \frac{4\mu_{22}}{\sigma_{XY}^2} - \frac{4\mu_{31}}{\sigma_{XY}\sigma_X^2} - \frac{4\mu_{13}}{\sigma_{XY}\sigma_Y^2} \right)$$

and $\mu_{ij} = \mathrm{E}[(X_1 - \mu_X)^i (Y_1 - \mu_Y)^j]$.

Part 2: We can prove by using Kelley et al. (2018) directly.

(*Appendices continue*)

Appendix C

Derivation of Optimal and Pilot Sample Sizes

Derivation of Optimal Sample Size for Squared Multiple Correlation Coefficient

For the situation in which the population squared multiple correlation coefficient is known, derivation of the optimal sample size is given as

$$\begin{aligned}
&\exp\left(\ln(1-R^2)+\frac{2Pz_{\alpha/2}}{\sqrt{n-k-2}}\right)-\exp\left(\ln(1-R^2)-\frac{2Pz_{\alpha/2}}{\sqrt{n-k-2}}\right)\le\omega\\
&(1-R^2)\left[\exp\left(\frac{2Pz_{\alpha/2}}{\sqrt{n-k-2}}\right)-\exp\left(\frac{-2Pz_{\alpha/2}}{\sqrt{n-k-2}}\right)\right]\le\omega\\
&\exp\left(\frac{2Pz_{\alpha/2}}{\sqrt{n-k-2}}\right)-\exp\left(\frac{-2Pz_{\alpha/2}}{\sqrt{n-k-2}}\right)\le\frac{\omega}{(1-R^2)}\\
&\exp\left(\frac{4Pz_{\alpha/2}}{\sqrt{n-k-2}}\right)-\frac{\omega}{(1-R^2)}\exp\left(\frac{2Pz_{\alpha/2}}{\sqrt{n-k-2}}\right)-1\le 0\\
&\frac{1}{2}\left(\frac{\omega}{(1-R^2)}+\sqrt{\frac{\omega^2}{(1-R^2)^2}+4}\right)\ge\exp\left(\frac{2Pz_{\alpha/2}}{\sqrt{n-k-2}}\right).
\end{aligned}\tag{84}$$

One may note that $\exp\left(\frac{2Pz_{\alpha/2}}{\sqrt{n-k-2}}\right)>0$. Thus, using Equation 84, we have

$$\begin{aligned}
&\frac{2Pz_{\alpha/2}}{\sqrt{n-k-2}}\le\ln\left[\frac{1}{2}\left(\frac{\omega}{(1-R^2)}+\sqrt{\frac{\omega^2}{(1-R^2)^2}+4}\right)\right]\\
&\sqrt{n-k-2}\ge 2Pz_{\alpha/2}\left\{\ln\left[\frac{1}{2}\left(\frac{\omega}{(1-R^2)}+\sqrt{\frac{\omega^2}{(1-R^2)^2}+4}\right)\right]\right\}^{-1}
\end{aligned}\tag{85}$$

$$n\ge 2+k+4P^2z_{\alpha/2}^2\left\{\ln\left[\frac{1}{2}\left(\frac{\omega}{(1-R^2)}+\sqrt{\frac{\omega^2}{(1-R^2)^2}+4}\right)\right]\right\}^{-2}.\tag{86}$$

Derivation of Pilot Sample Sizes

Derivation of pilot sample size m_{CC} for $r<1-\frac{\omega}{2}$ and $\omega<0.5$,

$$\begin{aligned}
&B=\frac{\left(1+|r|+\frac{\omega}{2}\right)\left(1-|r|+\frac{\omega}{2}\right)}{\left(1+|r|-\frac{\omega}{2}\right)\left(1-|r|-\frac{\omega}{2}\right)}<\frac{\left(2+\frac{\omega}{2}\right)\left(1+\frac{\omega}{2}\right)}{\left(1-\frac{\omega}{2}\right)\left(\frac{1-2\omega}{4}\right)}=b\\
&\Rightarrow\ln B<\ln b\quad\Rightarrow\quad\frac{1}{\ln B}>\frac{1}{\ln b}.
\end{aligned}\tag{87}$$

From the stopping rule in Equation 14 and using Equation 87, we have

$$\begin{aligned}
&n\ge 16z_{\alpha/2}^2\left(\frac{1}{(\ln B)^2}+\frac{1}{n}\right)+3\ge 16z_{\alpha/2}^2\left(\frac{1}{(\ln b)^2}+\frac{1}{n}\right)+3\\
&n^2\ge\left(\frac{16z_{\alpha/2}^2}{(\ln b)^2}+3\right)n+16z_{\alpha/2}^2\\
&n^2-\left(\frac{16z_{\alpha/2}^2}{(\ln b)^2}+3\right)n-16z_{\alpha/2}^2\ge 0.
\end{aligned}\tag{88}$$

Using quadratic formula and $n>0$, we have

$$n\ge\frac{1}{2}\left[3+\frac{16z_{\alpha/2}^2}{(\ln b)^2}+\sqrt{\left(3+\frac{16z_{\alpha/2}^2}{(\ln b)^2}\right)^2+(8z_{\alpha/2})^2}\right]\tag{89}$$

Thus, the pilot sample size m_{CC} is

$$m_{CC}=\max\left\{4,\left\lceil\frac{1}{2}\left[3+\frac{16z_{\alpha/2}^2}{(\ln b)^2}+\sqrt{\left(3+\frac{16z_{\alpha/2}^2}{(\ln b)^2}\right)^2+(8z_{\alpha/2})^2}\right]\right\rceil\right\}.\tag{90}$$

Derivation of pilot sample size $m_{MG.}$ from the stopping rule defined in Equation 19, we have

$$\begin{aligned}
&n\ge\frac{4z_{\alpha/2}^2}{\omega^2}\left[(1-r^2)^2+\frac{1}{n}\right]+1\ge\frac{4z_{\alpha/2}^2}{n\omega^2}+1\\
&n^2-n-\frac{4z_{\alpha/2}^2}{\omega^2}\ge 0.
\end{aligned}\tag{91}$$

Solving for n > 0, we have

$$n\ge\frac{1+\sqrt{1+(4z_{\alpha/2}/\omega)^2}}{2}.\tag{92}$$

The pilot sample size m_{MG} is therefore defined as

$$m_{MG}=\max\left\{4,\left\lceil\frac{1+\sqrt{1+(4z_{\alpha/2}/\omega)^2}}{2}\right\rceil\right\}.\tag{93}$$

Derivation of pilot sample size m_{BW} using the stopping rule in Equation 62,

$$\begin{aligned}
&n\ge 2+k+4R^2z_{\alpha/2}^2\left\{\ln\left[\frac{1}{2}\left(\frac{\omega}{(1-R^2)}+\sqrt{\frac{\omega^2}{(1-R^2)^2}+4}\right)\right]\right\}^{-2}\\
&\Rightarrow n\ge 2+k.
\end{aligned}\tag{94}$$

The pilot sample size m_{BW} is therefore defined as

$$m_{BW}=k+2.\tag{95}$$

Derivation of pilot sample size m_{OF}. Using the stopping rule in Equation 63, we have

$$\begin{aligned}
&n\ge\frac{16z_{\alpha/2}^2}{\omega^2}\left(R^2(1-R^2)^2+\frac{1}{n}\right)\ge\frac{16z_{\alpha/2}^2}{n\omega^2}\Rightarrow n^2\ge\frac{16z_{\alpha/2}^2}{\omega^2}\\
&\Rightarrow n\ge\frac{4z_{\alpha/2}}{\omega}.
\end{aligned}\tag{96}$$

The pilot sample size m_{OF} is therefore defined as

$$m_{OF}=\max\left\{k,\frac{4z_{\alpha/2}}{\omega}\right\}.\tag{97}$$

(*Appendices continue*)

Appendix D

Investigating Undercoverage for Some Simulation Conditions

We conducted another Monte Carlo simulation study using a fixed-*n* approach at the mean sample sizes obtained in Tables 1 and 2. The fixed-*n* approach using the mean final sample size from Tables 1 and 2 allow us to assess the extent to which the coverage issue is due to (a) our sequential AIPE procedure or (b) the distribution-free methods of confidence interval methods which depends on the estimation of the asymptotic variance. In particular, if the coverage probability from the fixed-*n* approach yielded good confidence interval coverage, then the under-coverage issue from Tables 1 and 2 can be attributed to the sequential AIPE procedure. However, if using a fixed-*n* approach with Equation 70 as an estimator of the asymptotic variance in Equation 3 yields the same under-coverage issues from Tables 1 and 2, then the under-coverage issue can be attributed to the confidence interval methods themselves.

Our results assessing the under-coverage issue are shown in Appendix E, which is based on 5,000 replications per condition.

We believe that the shortcoming in the confidence interval coverage is due to the bias of the estimator for each of the ratio of parameters given in Equation 3. Even though the individual parameters are estimated using unbiased estimators, the ratio of unbiased estimators cannot be said to be unbiased. Therefore, in order to get better results, a robust consistent estimator for the ratio of parameters may be used, but this is an active area of research and we are limited by what already exists in the literature. For example, the biased estimator of the kurtosis that is used in this article, along with five other estimators, was studied by An and Ahmed (2008).

Appendix E shows the average estimates of the terms in Equation 3 with their respective standard errors for different bivariate distributions and different sample sizes. Our results, based on 5,000 replications, show that most of the terms in Equation 3 are underestimated, especially for smaller sample sizes, with μ_{22}/σ^2_{XY} being overestimated at times. This is shown in Columns 6, 8, 10, 14, and 16 underestimating their respective theoretical values in Columns 5, 7, 9, 13, and 15. We also noticed that the average of the ratio of $\hat{\mu}_{22n}$ and S^2_{XYn}, $\overline{\hat{\mu}_{22n}/S^2_{XYn}}$, has a high standard error compared to the other terms in V^2_n (see Equation 70). These are as a results of very few outliers occurring when the estimated sample covariance S_{XYn} (for especially $\rho = 0.1$) is very small (i.e. close to zero). However, the few huge values of $\hat{\mu}_{22n}/S^2_{XYn}$ do not affect the estimate of V^2_n because if sample covariance S_{XYn} is close to zero, then so is sample correlation r. In other words, the inflation of $\hat{\mu}_{22n}/S^2_{XYn}$ is neutralized by the multiplicative factor r^2 as it can be seen in Equation 70.

This cumulatively results in the underestimation of ξ^2 (as shown in the last column). With ξ^2 being underestimated, the sequential procedure can terminate prematurely and thus result in underestimation of the minimum required sample size n. If the minimum required sample size is not achieved for the procedure to estimate the asymptotic variance appropriately, the confidence interval obtained cannot guarantee the desired coverage probability even though its length will not exceed the desired upper bound ω. We can say that the proposed sequential procedure works well under bivariate normal distribution but not under the selected bivariate gamma distribution. Thus, the performance of our procedure, under the bivariate gamma distribution and/or small sample sizes, is being negatively affected by the performance of the estimator of ξ^2. To increase the performance, efficiency, and robustness of the proposed sequential procedure, future studies can be conducted and aimed at improving the estimator of ξ^2.

(*Appendices continue*)

Appendix E

Simulation Results for Investigating the Performance of the Estimators of the Terms Involved in ξ^2

Gamma			$\bar{r}_n$		$\overline{\frac{\hat{\mu}_{40n}}{S_{Xn}^4}}$		$\overline{\frac{\hat{\mu}_{04n}}{S_{Yn}^4}}$		$\overline{\frac{\hat{\mu}_{22n}}{S_{Xn}^2 S_{Yn}^2}}$		$\overline{\frac{\hat{\mu}_{22n}}{S_{XYn}^2}}$		$\overline{\frac{\hat{\mu}_{31n}}{S_{XYn}S_{Xn}^2}}$		$\overline{\frac{\hat{\mu}_{13n}}{S_{XYn}S_{Yn}^2}}$		$\overline{V_n^2}$
Distribution	n	ρ	$se(\bar{r}_n)$	$\frac{\mu_{40}}{\sigma_X^4}$	$se\left(\overline{\frac{\hat{\mu}_{40n}}{S_{Xn}^4}}\right)$	$\frac{\mu_{04}}{\sigma_Y^4}$	$se\left(\overline{\frac{\hat{\mu}_{04n}}{S_{Yn}^4}}\right)$	$\frac{\mu_{22}}{\sigma_X^2\sigma_Y^2}$	$se\left(\overline{\frac{\hat{\mu}_{22n}}{S_{Xn}^2 S_{Yn}^2}}\right)$	$\frac{\mu_{22}}{\sigma_{XY}^2}$	$se\left(\overline{\frac{\hat{\mu}_{22n}}{S_{XYn}^2}}\right)$	$\frac{\mu_{31}}{\sigma_{XY}\sigma_X^2}$	$se\left(\overline{\frac{\hat{\mu}_{31n}}{S_{XYn}S_{Xn}^2}}\right)$	$\frac{\mu_{13}}{\sigma_{XY}\sigma_Y^2}$	$se\left(\overline{\frac{\hat{\mu}_{13n}}{S_{XYn}S_{Yn}^2}}\right)$	ξ^2	$se(\overline{V_n^2})$
(5, 5, 50, 10)	1124	.1	.10000 (.00042)	4.2	4.17166 (.00806)	4.2	4.16685 (.00814)	1.10825	1.11046 (.00197)	110.82353	207.66422 (19.73142)	4.2	4.11849 (.03202)	4.2	4.100631513 (.03403)	1.0508	1.04738 (.00153)
(5, 5, 16.67, 10)	1049	.3	.30048 (.00044)	4.2	4.17431 (.00878)	4.2	4.15289 (.00833)	1.48455	1.48165 (.00357)	16.49475	16.68271 (.04745)	4.2	4.16428 (.01335)	4.2	4.143475493 (.01291)	.9843	.97603 (.00163)
(5, 5, 10, 10)	774	.5	.49907 (.00044)	4.2	4.14773 (.00949)	4.2	4.15199 (.00957)	2.05455	2.03400 (.00583)	8.21818	8.17093 (.02106)	4.2	4.13840 (.01191)	4.2	4.145818738 (.01204)	.7364	.72802 (.00155)
(5, 5, 50, 10)	272	.1	.09986 (.00089)	4.2	4.08589 (.01521)	4.2	4.06454 (.01487)	1.10825	1.10894 (.00380)	110.82353	264979.14038 (227247.63189)	4.2	3.44243 (1.26490)	4.2	2.58627529 (1.95399)	1.0508	1.02837 (.00279)
(5, 5, 16.67, 10)	246	.3	.30006 (.00089)	4.2	4.04042 (.01479)	4.2	4.03521 (.01565)	1.48455	1.46366 (.00668)	16.49475	18.00356 (.25173)	4.2	4.00945 (.02384)	4.2	3.98244151 (.02437)	.9843	.95134 (.00296)
(5, 5, 10, 10)	164	.5	.49859 (.00094)	4.2	3.99107 (.01742)	4.2	4.01278 (.01729)	2.05455	2.00865 (.01079)	8.21818	8.15213 (.03865)	4.2	4.02036 (.02136)	4.2	4.033506793 (.02146)	.7364	.70087 (.00289)
(5, 5, 50, 10)	1600	.1	.09944 (.00036)	4.2	4.18424 (.00694)	4.2	4.18652 (.00714)	1.10825	1.10862 (.00167)	110.82353	160.16649 (13.94716)	4.2	4.17051 (.02469)	4.2	4.164455363 (.02537)	1.0508	1.04720 (.00131)
(5, 5, 16.67, 10)	1497	.3	.30020 (.00037)	4.2	4.17056 (.00713)	4.2	4.18356 (.00742)	1.48455	1.47906 (.00298)	16.49475	16.59995 (.03932)	4.2	4.15590 (.01086)	4.2	4.162168873 (.01137)	.9843	.97711 (.00138)
(5, 5, 10, 10)	1109	.5	.49905 (.00036)	4.2	4.17956 (.00823)	4.2	4.15990 (.00806)	2.05455	2.04565 (.00494)	8.21818	8.21482 (.01770)	4.2	4.17576 (.01029)	4.2	4.165810683 (.01013)	.7364	.73315 (.00132)
(5, 5, 50, 10)	391	.1	.09979 (.00074)	4.2	4.10117 (.01275)	4.2	4.12947 (.01300)	1.10825	1.10863 (.00325)	110.82353	1071921.82564 (998634.17011)	4.2	3.28202 (.86017)	4.2	2.274885338 (2.46682)	1.0508	1.03462 (.00245)
(5, 5, 16.67, 10)	360	.3	.30016 (.00074)	4.2	4.11350 (.01387)	4.2	4.09600 (.01309)	1.48455	1.48913 (.00637)	16.49475	17.43061 (.09181)	4.2	4.11317 (.02208)	4.2	4.114107163 (.02139)	.9843	.96767 (.00273)
(5, 5, 10, 10)	252	.5	.49885 (.00077)	4.2	4.05961 (.01544)	4.2	4.06889 (.01497)	2.05455	2.01381 (.00918)	8.21818	8.12197 (.03228)	4.2	4.06269 (.01884)	4.2	4.061168915 (.01828)	.7364	.71160939 (.00242)

Note. n is the sample size used in the simulation (and the values were taken from $\bar{N}$ column of Tables 1 and 2); ρ is the population correlation coefficient; $\bar{r}_n$ and $se(\bar{r}_n)$ are the average correlation coefficient and its standard error respectively; ξ^2 is population asymptotic variance shown in Equation 3; $\overline{\frac{\hat{\mu}_{40n}}{S_{Xn}^4}}$ is the mean estimate of the population value $\frac{\mu_{40}}{\sigma_X^4}$ and its standard error is $se\left(\overline{\frac{\hat{\mu}_{40n}}{S_{Xn}^4}}\right)$; $\overline{\frac{\hat{\mu}_{04n}}{S_{Yn}^4}}$ is the mean estimate of the population value $\frac{\mu_{04}}{\sigma_Y^4}$ and its standard error is $se\left(\overline{\frac{\hat{\mu}_{04n}}{S_{Yn}^4}}\right)$; $\overline{\frac{\hat{\mu}_{22n}}{S_{Xn}^2 S_{Yn}^2}}$ is the mean estimate of the population value $\frac{\mu_{22}}{\sigma_X^2\sigma_Y^2}$ and its standard error is $se\left(\overline{\frac{\hat{\mu}_{22n}}{S_{Xn}^2 S_{Yn}^2}}\right)$; $\overline{\frac{\hat{\mu}_{22n}}{S_{XYn}^2}}$ is the mean estimate of the population value $\frac{\mu_{22}}{\sigma_{XY}^2}$ and its standard error is $se\left(\overline{\frac{\hat{\mu}_{22n}}{S_{XYn}^2}}\right)$; $\overline{\frac{\hat{\mu}_{31n}}{S_{XYn}S_{Xn}^2}}$ is the mean estimate of the population value $\frac{\mu_{31}}{\sigma_{XY}\sigma_X^2}$ and its standard error is $se\left(\overline{\frac{\hat{\mu}_{31n}}{S_{XYn}S_{Xn}^2}}\right)$; $\overline{\frac{\hat{\mu}_{13n}}{S_{XYn}S_{Yn}^2}}$ is the mean estimate of the population value $\frac{\mu_{13}}{\sigma_{XY}\sigma_Y^2}$ and its standard error is $se\left(\overline{\frac{\hat{\mu}_{13n}}{S_{XYn}S_{Yn}^2}}\right)$; $\overline{V_n^2}$ is the average estimate of ξ^2 and its standard error is $se(\overline{V_n^2})$.

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